Stability Analysis of Swarms

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Abstract

In this brief article we specify an “individual-based” continuous time model for swarm aggregation in $n$-dimensional space and study its stability properties. We show that the individuals (autonomous agents or biological creatures) will form a cohesive swarm in a finite time. Moreover, we obtain an explicit bound on the swarm size, which depends only on the parameters of the swarm model.

I. INTRODUCTION

For a long time it has been observed that certain living beings tend to perform swarming behavior. Examples of swarms include the flocks of birds, schools of fish, herds of animals, and colonies of bacteria. It is known that such a cooperative behavior has certain advantages such as avoiding predators and increasing the chance of finding food but it requires communications and coordinated decision making. Operational principles from such systems can be used in engineering for developing distributed cooperative control, coordination, and learning strategies for autonomous agent systems such as autonomous multi-robot applications, unmanned undersea, land, or air vehicles. There are, however, several key steps to exploit biological principles to develop such highly automated systems. These include modeling, coordination strategy specification, and analysis to show that group dynamics achieve group goals. In this article we develop a simple model describing swarm aggregation and analyze its stability properties. We show that the individuals will form a cohesive swarm in a finite time. Moreover, we obtain a bound on the swarm size, which depends only on the parameters of the swarm model.

Biologists have been working on understanding and modeling of swarming behavior for a long time [1], [2], [3], [4]. The general understanding now is that the swarming behavior is a result of an interplay between a long range attraction and a short range repulsion between the individuals. In [1] Breder suggested a simple model composed of a constant attraction term and a repulsion term which is inversely proportional to the square of the distance between two members, whereas in [2] Warburton and Lazarus studied the affect on cohesion of a family of attraction/repulsion functions. The articles in [3] and [4] provide good background and review of the swarm modeling concepts and literature such as spatial and nonspatial models, individual-based versus continuum models and so on. See also [5] and references therein for other related work.

In parallel to the mathematical biologists there are a number of physicists who have done
important work on swarming behavior [6], [7], [8], [9], [10], [11]. The general approach the physicists take is to model each individual as a particle and study the collective behavior due to their interaction. Many of them assume that particles are moving with constant absolute velocity and at each time step assume the average direction of motion of the particles in its neighborhood with some random perturbation. They try to study the affect of the noise on the collective behavior and to validate their models through extensive simulations.

In recent years, engineering applications such as formation control of multirobot teams and autonomous air vehicles have emerged and this has increased the interest of engineers in swarms. Some examples include [12] and [13], where the authors describe formation control strategies for autonomous air vehicles and multiple autonomous land vehicle teams, respectively. In [14] Reif and Wang consider distributed control approach of groups of robots, called social potential fields method, which is based on artificial force laws between individual robots and robot groups. The force laws are inverse-power or spring force laws incorporating both attraction and repulsion. Another work on distributed formation control of robots is [15], where the authors consider asynchronous distributed control and geometric pattern formation of multiple anonymous (or identical) robots.

Important work on swarm stability is given by Beni and coworkers in [16] and [17]. In [16] they consider a synchronous distributed control method for discrete one and two dimensional swarm structures and prove stability in the presence of disturbances using Lyapunov methods. On the other hand, [17] is, to best of our knowledge, one of the first stability results for asynchronous methods (with no time delays). There they consider a linear swarm model and prove sufficient conditions for the asynchronous convergence of the swarm to a synchronously achievable configuration.

Swarm stability under total asynchronism (i.e., asynchronism with time delays) was first considered in [18], [19], [20]. In [18] a one dimensional discrete time totally asynchronous swarm model is proposed and stability (swarm cohesion) is proved. The authors prove asymptotic convergence under total asynchronism conditions and finite time convergence under partial asynchronism conditions (i.e., total asynchronism with a bound on the maximum possible time delay). In [19], on the other hand, the authors consider a mobile swarm model and prove that cohesion will be preserved during motion under certain conditions, expressed as bounds on the
maximum possible time delay.

In [21] we obtained similar results to those in [18] for a swarm with a different mathematical model for the intermember interactions and motions using some earlier results developed for parallel and distributed computation in computer networks in [22].

All of these stability investigations have been limited to either one or two dimensional space. Note that in one dimension, the problem of swarming is very similar to the problem of *platooning* of vehicles in *automated highway systems*, an area that has been studied extensively (see, for example, [23], [24], [25] and references therein).

II. A Model of an Aggregating Swarm

Consider a swarm of $M$ individuals (members) in an $n$-dimensional Euclidian space. We model the individuals as points and ignore their dimensions. The position of member $i$ of the swarm is described by $x_i \in \mathbb{R}^n$. We assume synchronous motion and no time delays, i.e., all the members move simultaneously and know the exact position of all the other members. The motion dynamics evolve in continuous time. The equation of motion that we consider for individual $i$ is given by

$$\dot{x}_i = \sum_{j=1, j \neq i}^{M} g(x_j - x_i), i = 1, \ldots, M,$$

where $g(\cdot)$ represents the function of attraction and repulsion between the members. In other words, the direction and magnitude of motion of each member is determined as a sum of the attraction and repulsion of all the other members on this member. The attraction/repulsion function that we consider is

$$g(y) = -y \left( a - b \exp \left( -\frac{\|y\|^2}{c} \right) \right),$$

where $a$, $b$, and $c$ are positive constants and $\|y\| = \sqrt{y^t y}$. For the $y \in \mathbb{R}^1$ case with $a = 1$, $b = 20$, and $c = 0.2$ this function is shown in Figure 1. In higher dimensions (i.e., $y \in \mathbb{R}^n$), the function is exactly the same as in one dimensional case, except that it acts on the line connecting the positions of the two members (i.e., the line on which the vector $y$ lies).

Note that this function is attractive for large distances and repulsive for small distances. By equating $y \left( a - b \exp \left( -\frac{\|y\|^2}{c} \right) \right) = 0$, one can easily find that $g(y)$ switches sign at the set of
points defined as
\[ \mathcal{Y} = \left\{ y = 0 \text{ or } \|y\| = \delta = \sqrt{c \ln \left( \frac{b}{a} \right)} \right\} . \]

Notice that implicitly it is assumed that \( b > a \), since otherwise the expression will never switch sign except at zero and there will not be any repulsion between the members no matter how close they are to each other.

Define the center of the swarm members as \( \bar{x} = \frac{1}{M} \sum_{i=1}^{M} x^i \). Note that because of the symmetry of \( g(\cdot) \) the center \( \bar{x} \) is stationary for all \( t \). In other words, since \( g(\cdot) \) is symmetric with respect to the origin, member \( i \) moves toward every other member \( j \) exactly the same amount as \( j \) moves toward \( i \). We express this more formally in the following lemma.

**Lemma 1**: The center \( \bar{x} \) of the swarm described by the model in Eq. (1) with an attraction/repulsion function \( g(\cdot) \) as given in Eq. (2) is stationary for all \( t \).

**Proof**: Note that
\[
\dot{\bar{x}} = \frac{1}{M} \sum_{i=1}^{M} \dot{x}^i
= -\frac{1}{M} \sum_{i=1}^{M} \sum_{j=1, j \neq i}^{M} (x^i - x^j) \left[ a - b \exp \left( -\frac{\|x^i - x^j\|}{c} \right) \right]
= -\frac{1}{M} \sum_{i=1}^{M} x^i \sum_{j=1}^{M} \left[ a - b \exp \left( -\frac{\|x^i - x^j\|}{c} \right) \right] - \sum_{j=1}^{M} x^j \sum_{i=1}^{M} \left[ a - b \exp \left( -\frac{\|x^i - x^j\|}{c} \right) \right]
= 0.
\]
Basically this lemma says that, on average, the swarm described by Eq. (1) with an attraction/repulsion function as given in Eq. (2) is not drifting. Note, however, that although it states that the center of the swarm is stationary, it does not say anything about the relative motions of the members with respect to it. It may be the case that the members diverge from the center while it stays stationary. Intuitively, however, we would expect the members to move toward the center for the given swarm model. In several of the results and discussions to follow we either implicitly or explicitly will use the fact that $\bar{x}$ is stationary.

III. ANALYSIS OF SWARM COHESION

Our first result is about a swarm member which does not have any neighbors in its repulsion range. We call such a member a free agent.

Definition 1: A swarm member $i$ is called a free agent if

$$\|x^i - x^j\| > \delta, \forall j \in S, j \neq i,$$

where $S = \{1, \ldots, M\}$ is the set of members of the swarm.

Note that since the distance from all the other members to a free agent is greater than $\delta$, there will not be any repulsion force and the total force on this member will be a combined effect of all the attraction imposed by all the other members. We will show that this force is pointing toward the center $\bar{x}$ of the swarm, and therefore, the member is moving toward it. Before stating this result more rigorously, we define the error variable as

$$e^i = x^i - \bar{x},$$

for each individual $i = 1, \ldots, M$.

Lemma 2: Assume that a member $i$ of the swarm described by the model in Eq. (1) with an attraction/repulsion function $g(\cdot)$ as given in Eq. (2) is a free agent at time $t$ and that its distance to the center $\bar{x}$ of the swarm is greater then $\delta$, i.e.,

$$\|e^i(t)\| = \|x^i(t) - \bar{x}\| > \delta.$$

Then, at time $t$ its motion is in a direction of decrease of $\|e^i(t)\|$ (i.e., toward the center $\bar{x}$).
Proof: From the definition of the center $\bar{x}$ of the swarm we have $\sum_{j=1}^{M} (x^j - x^i) = M(\bar{x} - x^i)$. Subtracting from both sides $Mx^i$ we obtain

$$\sum_{j=1}^{M} (x^j - x^i) = M(x^i - \bar{x}) = Me^i. \quad (3)$$

Then, the motion of member $i$ can be represented as

$$\dot{x}^i = \sum_{j=1, j \neq i}^{M} g(x^j - x^i)$$

$$= -a \sum_{j=1, j \neq i}^{M} (x^j - x^i) + b \sum_{j=1, j \neq i}^{M} \exp\left(-\frac{\|x^j - x^i\|^2}{c}\right)(x^j - x^i)$$

$$= -aMe^i + b \sum_{j=1, j \neq i}^{M} \exp\left(-\frac{\|x^j - x^i\|^2}{c}\right)(x^j - x^i),$$

where on the second line we used the definition of function $g(\cdot)$ in Eq. (2) and added $a(x^j - x^i) = 0$, and substituted the value of $\sum_{j=1}^{M} (x^j - x^i)$ from Eq. (3) on the third.

Note that since $\dot{x} = 0$, we have $\dot{e}^i = \dot{x}^i$. Choosing the Lyapunov function candidate for member $i$ as

$$V_i = \frac{1}{2}e^{T}e^i$$

and taking its derivative along the trajectory of the member we obtain

$$\dot{V}_i = \dot{e}^i e^i = -aM\|e^i\|^2 + \sum_{j=1, j \neq i}^{M} b \exp\left(-\frac{\|x^j - x^i\|^2}{c}\right)(x^j - x^i)^{\top}e^i. \quad (4)$$

Note that the expression $b \exp\left(-\frac{\|x^j - x^i\|^2}{c}\right) > 0$ for all $x^j$ and $x^i$. Therefore, $\dot{V}_i$ is bounded by

$$\dot{V}_i \leq -aM\|e^i\|^2 + \sum_{j=1, j \neq i}^{M} b \exp\left(-\frac{\|x^j - x^i\|^2}{c}\right)\|x^j - x^i\|\|e^i\|. \quad (5)$$

Since member $i$ is a free agent, we have $\|x^j - x^i\| > \delta, \forall j \neq i$ and note that for that range the function $\exp\left(-\frac{\|x^j - x^i\|^2}{c}\right)\|x^j - x^i\|$ is a decreasing function of the distance with the maximum occurring at $\|x^j - x^i\| = \delta$. Using these facts, we have

$$\dot{V}_i \leq -aM\|e^i\|^2 + b(M-1)\delta \exp\left(-\frac{\delta^2}{c}\right)\|e^i\|$$

$$= -a\|e^i\|^2 - (M-1) \left[a\|e^i\| - b\delta \exp\left(-\frac{\delta^2}{c}\right)\right]\|e^i\|. \quad (6)$$

$$= -a\|e^i\|^2 - (M-1) \left[a\|e^i\| - b\delta \exp\left(-\frac{\delta^2}{c}\right)\right]\|e^i\|. \quad (7)$$
For the second term to be negative semidefinite we need
\[ \|e^i\| \geq \frac{b\delta}{a}\exp\left(-\frac{\delta^2}{c}\right). \]

Note, however, that \( \frac{b}{a}\exp\left(-\frac{\delta^2}{c}\right) = 1 \), which implies that we need \( \|e^i\| \geq \delta \), which, on the other hand, holds by our hypothesis. Therefore, we have
\[ \dot{V}_i \leq -a\|e^i\|^2 = -2aV_i, \]
which proves the assertion.

**Remark:** From attraction/repulsion function \( g(\cdot) \) in Eq. (2) one can see that one term in \( g(\cdot) \) always gives attraction and the other repulsion and the resultant force is their sum. This leads to similar terms in the derivative of the Lyapunov function in Eq. (4). If an individual is away from all the other individuals, the second term in the Lyapunov function is negligibly small compared to the first term and it moves toward the center. If it is close to the other individuals (i.e., in their repulsion range), then the second term becomes significant.

Note that Lemma 2 does not imply that \( x^i \) will converge to \( \bar{x} \) for all \( i \). Intuitively, once a member gets to the vicinity of another member, then the repulsive force will be in effect and the conditions of Lemma 2 will not be satisfied anymore. However, it is important because it gives us an idea of the tendency of the individuals to move toward the center of the swarm. Therefore, it is normal to expect that the members will (potentially) aggregate and form a cluster around \( \bar{x} \). To prove this we need to analyze the motion of the members which are not necessarily free agents and that is done in the next result.

**Theorem 1:** Consider the swarm described by the model in Eq. (1) with an attraction/repulsion function \( g(\cdot) \) as given in Eq. (2). As time progresses all the members of the swarm will converge to a hyperball
\[ B_\varepsilon(x) = \{ x : \|x - \bar{x}\| \leq \varepsilon \}, \]
where
\[ \varepsilon = \frac{b}{a}\sqrt{\frac{c}{2}}\exp\left(-\frac{1}{2}\right). \]
Moreover, the convergence will occur in finite time bounded by
\[ \bar{t} = \max_{i \in S} \left\{ -\frac{1}{2a} \ln \left( \frac{\varepsilon^2}{2V_i(0)} \right) \right\}. \]
Proof: Choose any swarm member $i$. Let $V_i = \frac{1}{2} e_i^T e^i$ be the corresponding Lyapunov function. From the proof of Lemma 2 we know that

$$\dot{V}_i = -aM\|e^i\|^2 + \sum_{j=1, j\neq i}^{M} b \exp\left(-\frac{\|x^i - x^j\|^2}{c}\right) (x^i - x^j)^T e^i.$$  \hspace{1cm} (8)

Therefore, if

$$\|e^i\| > \frac{b}{aM} \sum_{j=1, j\neq i}^{M} \exp\left(-\frac{\|x^i - x^j\|^2}{c}\right) \|x^i - x^j\|,$$

then we will have $\dot{V}_i < 0$.

This bound is a function of the distance between the members. Note that each function in the sum on the right hand side is a bounded function and by using its maximum we can obtain a position independent bound. Solving for the maximum (i.e., solving the equation $\frac{\partial}{\partial y} \left( y \exp\left(-\frac{y^2}{c}\right) \right) = \exp\left(-\frac{y^2}{c}\right) - 2\frac{y^2}{c} \exp\left(-\frac{y^2}{c}\right) = 0$) we obtain that it occurs at $\|x^i - x^j\| = \sqrt{\frac{c}{2}}$, or in other words, the maximum occurs when the members are at a distance $\sqrt{\frac{c}{2}}$ from each other. Evaluating the maximum we have $\sqrt{\frac{c}{2}} \exp\left(-\frac{(\sqrt{\frac{c}{2}})^2}{c}\right) = \sqrt{\frac{c}{2}} \exp\left(-\frac{1}{2}\right)$. Substituting this in the above equation we obtain that $\dot{V}_i < 0$ as long as

$$\|e^i\| > \frac{b(M-1)}{aM} \sqrt{\frac{c}{2}} \exp\left(-\frac{1}{2}\right).$$

Define

$$\varepsilon = \frac{b}{a} \sqrt{\frac{c}{2}} \exp\left(-\frac{1}{2}\right)$$

and note that $\varepsilon > \frac{b(M-1)}{aM} \sqrt{\frac{c}{2}} \exp\left(-\frac{1}{2}\right)$. This implies that as $t \to \infty$, $e^i$ converges within the ball around $\bar{x}$ defined by $\frac{b(M-1)}{aM} \sqrt{\frac{c}{2}} \exp\left(-\frac{1}{2}\right)$. Since $\varepsilon > \frac{b(M-1)}{aM} \sqrt{\frac{c}{2}} \exp\left(-\frac{1}{2}\right)$ we have $e^i \to B\varepsilon$.

Since member $i$ was an arbitrary member, the result holds for all the members. To prove the finite time convergence note that the equation of $\dot{V}_i$ can be written as

$$\dot{V}_i \leq -a\|e^i\|^2 - a(M-1)\|e^i\|^2 \left[ \|e^i\| - \frac{b}{a(M-1)} \sum_{j=1, j\neq i}^{M} \exp\left(-\frac{\|x^i - x^j\|^2}{c}\right) \|x^i - x^j\| \right],$$

which implies that for $\|e^i\| \geq \varepsilon$, we have

$$\dot{V}_i \leq -a\|e^i\|^2 = -2aV_i.$$
Therefore, the solution of $V_i$ satisfies

$$V_i(t) \leq V_i(0)e^{-2at}.$$  

For $\|e^i\| \geq \varepsilon$ we have $V_i = \frac{1}{2}\varepsilon^2$ and individual $i$ enters the $\varepsilon$ vicinity of $\bar{x}$ at time $t_i$ when the right hand side of the above equation satisfies

$$V_i(0)e^{-2at_i} = \frac{1}{2}\varepsilon^2.$$  

Solving for $t_i$ we obtain

$$t_i \leq -\frac{1}{2a} \ln \left( \frac{\varepsilon^2}{2V_i(0)} \right),$$

and this proves the theorem.  

This result is important not only because it proves the cohesiveness of the swarm, but also it provides an explicit bound on the size of the swarm. Note that the bound $\varepsilon$ makes intuitive sense. To see this note that increasing parameter $a$ (i.e., increasing attraction) decreases the size of the bound $\varepsilon$. In contrast, increasing parameter $b$ (i.e., increasing repulsion magnitude) or parameter $c$ (increasing repulsion range) increases $\varepsilon$ and these are intuitively expected results. For the $g(\cdot)$ function given in Figure 1 with parameters $a = 1$, $b = 20$, and $c = 0.2$, we have $\varepsilon \approx 3.8$.

**Remark:** Note that the bound on the swarm size $\frac{b|M-1|}{aM} \sqrt{2} \exp \left(-\frac{1}{2}\right)$ depends on $M$. Therefore, for swarms with a small number of members the bound will differ significantly for different values of $M$. However, in biological swarms the number of the members $M$ can be very large and as $M \to \infty$ we have $\frac{b|M-1|}{aM} \sqrt{2} \exp \left(-\frac{1}{2}\right) \to \varepsilon$. In other words, $\varepsilon$ is the maximum possible bound on the swarm size independent of the number of the individuals in the swarm.

**Remark:** In view of the above remark, for large values of $M$ the size of the cohesive swarm is relatively independent of the number of the members (individuals). In other words, it is almost constant independent of the number of the members. This implies that as the number of the members increases the density of the swarm will also increase. This is inconsistent with some biological examples and is due to the particular attraction/repulsion function $g(\cdot)$ that we chose.

**Remark:** Note that even the bound $\frac{b(M-1)}{aM} \sqrt{2} \exp \left(-\frac{1}{2}\right)$ is very conservative, because above we used $(x^i - x^j)^\top e^i \leq \|x^i - x^j\|\|e^i\|$ and also assumed that the functions $\exp \left(-\frac{\|x^i - x^j\|^2}{c^2}\right)$
\(x^i\) are at their peak values for all \(i\) and \(j\) and these both are never the case. Therefore, the actual size of the swarm is, in general, much smaller than \(\epsilon\).

IV. ANALYSIS OF SWARM MEMBER BEHAVIOR IN A COHESIVE SWARM

Theorem 1 shows only the region where the swarm members will converge and provides a bound on the size of the swarm. It does not, however, say anything about whether the swarm members will stop their motion or will start an oscillatory motion within the region and this issue needs to be investigated further. To this end, first, we define the state \(x\) of the system as the vector of the positions of the swarm members \(x = [x^1, \ldots, x^M]^\top\). Let the invariant set of equilibrium points be

\[\Omega_e = \{x : \dot{x} = 0\}\]

We will prove that as \(t \to \infty\) the state \(x(t)\) converges to \(\Omega_e\), i.e., the configuration of the swarm members converges to a constant arrangement.

**Theorem 2:** Consider the swarm described by the model in Eq. (1) with an attraction/repulsion function \(g(\cdot)\) as given in Eq. (2). As \(t \to \infty\) we have \(x(t) \to \Omega_e\).

**Proof:** We choose the Lyapunov function

\[J(x) = \frac{1}{2} \sum_{i=1}^{M-1} \sum_{j=i+1}^{M} \left[ a \|x^i - x^j\|^2 + bc \exp \left( -\frac{\|x^i - x^j\|^2}{c} \right) \right].\]

Then, the gradient of \(J(x)\) with respect to each \(x^i\) is given by

\[
\nabla_{x^i} J(x) = \sum_{j=1, j \neq i}^{M} (x^j - x^i) \left[ a - b \exp \left( -\frac{\|x^i - x^j\|^2}{c} \right) \right]
= -\sum_{j=1, j \neq i}^{M} g(x^j - x^i) = -\dot{x}^i.
\]

Now, taking the time derivative of the Lyapunov function along the motion of the system we obtain

\[
\dot{J}(x) = [\nabla_{x^i} J(x)]^\top \dot{x} = \sum_{i=1}^{M} [\nabla_{x^i} J(x)]^\top \dot{x}^i = \sum_{i=1}^{M} [-\dot{x}^i]^\top \dot{x}^i = -\sum_{i=1}^{M} \|\dot{x}^i\|^2 \leq 0,
\]

for all \(t\). Then, using the LaSalle’s Invariance Principle we conclude that as \(t \to \infty\) the state \(x\) converges to the largest invariant subset of the set defined as

\[\Omega = \{x : \dot{J}(x) = 0\} = \{x : \dot{x} = 0\} = \Omega_e.\]
Since each point in $\Omega_e$ is an equilibrium, $\Omega_e$ is an invariant set and this proves the result.

**Remark:** The proof of the above theorem shows the distributed aspect of the swarming behavior. In fact, it shows that the swarm members are performing *distributed optimization* (function minimization) of a common function (the Lyapunov or cost function) using a *distributed gradient method*. In other words, each member computes its part of the gradient of the global function at its position (i.e., computes the gradient with respect to its motion variables) and moves along the negative direction of that gradient. The global function in this case is a function of the distances between the members. However, the idea could be transferred to the more general case in which any general global cost function could be considered.

**Remark:** Another view on the distributed nature of the approach can be as follows. Define

$$J_i(x) = \frac{1}{2} \sum_{j=1, j \neq i}^M \left[ a \| x_i^j - x_i^j \|^2 + bc \exp \left( \frac{\| x_i^j - x_i^j \|^2}{c} \right) \right].$$

Then, note that

$$\dot{x}_i^j = -\nabla_{x_i} J_i(x) = -\nabla_{x_i} J(x).$$

This can be interpreted as each member $i$ performing an optimization of its local cost function $J_i(x)$, which results in minimizing of the combined cost function

$$J(x) = \frac{1}{2} \sum_{i=1}^M J_i(x)$$

to obtain the overall behavior of the swarm.

**Remark:** The combination of the above results (Theorems 1 and 2) prove that the swarm described by the model in Eq. (1) with an attraction/repulsion function $g(\cdot)$ as given in Eq. (2) will be cohesive and also that the members will converge to a constant position. To our knowledge no such results exist in the literature for the type of model we use.

**Remark:** Note that in any of the above analysis we did not use the dimension of the state space $n$. Therefore, the results obtained hold for any dimension $n$.

**Remark:** The results here are global. This is a consequence of the definition of the attraction/repulsion function $g(\cdot)$ in Eq. (2) over the entire domain.
REFERENCES


