International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.informaworld.com/smpp/title~content=t713393989

Stability analysis of load balancing systems

Kevin L. Burgess [a]; Kevin M. Passino [a]
[a] Department of Electrical Engineering, Ohio State University, Columbus, OH, U.S.A.

Online Publication Date: 01 February 1995
To cite this Article: Burgess, Kevin L. and Passino, Kevin M. (1995) 'Stability analysis of load balancing systems', International Journal of Control, 61:2, 357 - 393
To link to this article: DOI: 10.1080/00207179508921907
URL: http://dx.doi.org/10.1080/00207179508921907

INT. J. CONTROL, 1995, VOL. 61, NO. 2, 357–393

Stability analysis of load balancing systems

KEVIN L. BURGESS† and KEVIN M. PASSINO†‡

A *load processor* is a system that has a buffer which can receive load and store it while it is waiting to be processed and has a local decision-making policy for determining if portions of its load should be sent to other load processors. A *load balancing system* is a set of such load processors that are connected in a network so that (i) they can sense the amount of load in the buffers of neighbouring processors and pass load to them, and (ii) so that, via local information and decisions by the individual load processors, the overall load in the entire network can be balanced. Such balancing is important to ensure that certain processors are not overloaded while others are left idle (i.e. load balancing helps avoid underutilization of processing resources). The topology of the network, delays in transporting and sensing load, types of load, and types of local load passing policies all affect the performance of the load balancing system. In this paper, we show how a variety of load balancing systems can be modelled in a discrete event system (DES) theoretic framework, and how balancing properties and performance can be characterized and analysed in a general Lyapunov stability theoretic framework.

1. Introduction

A load balancing system is a network of load processors (e.g. machines in a manufacturing system, computers on a network) that are connected together so that any processor on the network is capable of passing a portion of its load (e.g. jobs, tasks, parts) to any other processor to which it is connected, and if a processor can pass load to another load processor it can also sense the load level of that processor. Figure 1 illustrates an example load balancing system where

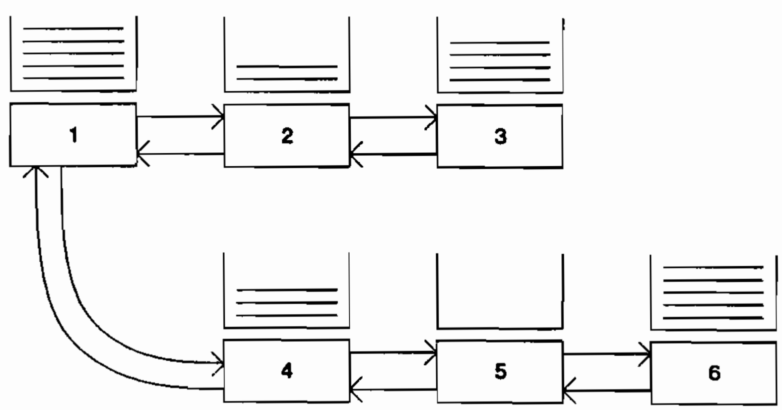

Figure 1. Example of a load balancing system.

Received 26 January 1993. Revised 13 December 1993.

† Department of Electrical Engineering, Ohio State University, 2015 Neil Avenue, Columbus, OH 43210, U.S.A.

‡ Tel: +(1) (614) 292-5716; e-mail: passino@ee.eng.ohio.state.edu.

each load processor along with its buffer is numbered from 1 to 6 and the arc from 1 to 2 indicates that 1 can sense the amount of load in the buffer of processor 2 and that 1 can pass load to processor 2. Since there are no arcs between 1 and 5, these processors cannot sense each other's loads or pass load to each other.

We are interested in studying the ability of the processors in such a system to redistribute the total network load so that it is balanced among all of the processors in the network (i.e. so the buffer levels are balanced). There may be delays incurred in the transportation of load from node to node and in the sensing of load between nodes. We assume that these delays are bounded. The load in the network may be such that it is valid to describe it with a continuous variable ('fluid load'), or it may be such that it can only be described as existing in discrete blocks (possibly of non-uniform size). Furthermore, the network operates in an asynchronous fashion so that each processor can decide when it wants to pass load independently of the others (moreover, allowing for various types of asynchronism ensures that the load balancing policies can be implemented in real-time).

In this analysis, which is based on the results presented by Burgess (1992), we model several variations of the load balancing problem in the DES framework of Passino *et al.* (1990, 1991, 1994) and analyse them via a Lyapunov stability approach. In particular, we will provide conditions under which the various load balancing systems are stable in the sense of Lyapunov, asymptotically stable, and exponentially stable, thereby characterizing the performance of the system's load redistribution policies. While it is possible to characterize and analyse certain 'stability' properties of DES with automata models and graph algorithms or with temporal logic and proof systems (see the references in Passino *et al.* 1990, 1991, 1994), in this work we investigate the characterization and analysis of conventional stability properties within a Lyapunov framework. In this way, we avoid the use of custom stability definitions and exploit several advantages of Lyapunov theory by showing that (i) it is possible to pick physically motivated Lyapunov functions that provide insight into the dynamics of how load balancing systems operate, (ii) by using the analytical Lyapunov approach we avoid having to enumerate all possible sequences of load transfers in showing that the system satisfies certain qualitative balancing properties, and (iii) a useful by-product of the Lyapunov analysis is obtained for those systems that can be shown to be exponentially stable (i.e. we provide a characterization of the 'speed' of balancing). To gain a full appreciation of the significance of the stability analysis in this paper and the wide number of applications where load balancing problems are encountered, the reader is referred to Shivaratri *et al.* (1992).

The load balancing systems that we examine are similar to, and generalizations of, those analysed by Passino *et al.* (1991) and Tsitsiklis and Bertsekas (1989). In Passino *et al.* (1990, 1991) the load balancing system is very simple because the load is considered to exist only in blocks of unit size, the allowed inter-processor load exchanges are quite restricted and any delays that exist in passing load and sensing load levels are ignored. The model of Tsitsiklis and Bertsekas (1989) assumes that load can be represented by a continuous variable and that delays exist in load passing and sensing. The model also allows for general load passing. Tsitsiklis and Bertsekas (1989) show that eventually the load will be perfectly balanced among the processors, and they suggest a proof for the 'geometric convergence' of the network to a balanced state.

We model the load balancing problem in the DES framework of Passino *et al.* (1990, 1991, 1994) and analyse it via the Lyapunov approach. Initially, we do not consider delays (in load passing and sensing) in our analysis. We show that non-delay systems are asymptotically stable under weaker passing conditions than in Tsitsiklis and Bertsekas (1989). We also show that under passing conditions similar to those in Tsitsiklis and Bertsekas (1989) that the non-delay load balancing system is exponentially stable. Additionally, we perform a rate of convergence analysis. We present generalized load passing conditions. We introduce the idea of 'virtual load' and demonstrate the asymptotic and exponential stability of non-delay virtual load systems. As a further generalization of the non-delay case, we prove asymptotic and exponential stability of systems in which the load is divided into discrete blocks of non-uniform size. In addition, we provide a rate of convergence analysis for the discrete load case.

We also study the full delay load balancing system as described by Tsitsiklis and Bertsekas (1989). Tsitsiklis and Bertsekas (1989) presented a proof for asymptotic stability and suggested a proof for geometric convergence. We take a different approach by studying the problem within the Lyapunov stability framework, proving asymptotic and exponential stability (in a different way), and providing a rate of convergence analysis. Three of the lemmas in our proof of exponential stability are adaptations of lemmas from the proof of Tsitsiklis and Bertsekas (1989). In order to use the exponential stability results of Michel *et al.* (1992 a, b) in our analysis, we generalize the conditions for exponential stability. The generalization allows us to prove exponential stability for the general delay case. As with the non-delay case, our exponential stability analysis helps to show how to provide a detailed characterization of the speed of balancing which can be obtained from a particular load balancing system.

Finally, we note that the load balancing problem considered by Cybenko (1994) is a special case of the one of Tsitsiklis and Bertsekas (1989) and the problems considered here. Passino and Antsaklis (1993) studied how to minimize the number of load transfers to achieve balancing by using global information about the load distribution in the system. In all the problems considered here, the load processors only use local information, balancing proceeds in an asynchronous fashion, and we do not consider trying to minimize the number of load transfers to achieve balancing. Also, note that while Boel and van Schuppen (1989) consider load balancing in a stochastic framework, our framework only admits deterministic load balancing problems.

In the next two subsections we introduce (i) the DES model we will use to represent the load balancing systems and (ii) the stability definitions and theorems that we use to characterize and analyse load balancing properties.

1.1. *A DES model*

We study the stability of systems that can be modelled via $G = (\mathcal{X}, \mathcal{E}, f_e, g, \boldsymbol{E}_v)$. $\mathcal{X}$ is the set of states and $\mathcal{E}$ is the set of events. State transitions are defined by the operators, $f_e: \mathcal{X} \rightarrow \mathcal{X}$ where $e \in \mathcal{E}$. An event, e, may only occur if it is in the set defined by the enable function, $g: \mathcal{X} \rightarrow \mathcal{P}(\mathcal{E}) - \{\phi\}$, where $\mathcal{P}(\mathcal{E})$ denotes the power set of $\mathcal{E}$. We only require that f_e be defined when $e \in g(\boldsymbol{x})$.

Notice that according to the definition of g, it can never be the case that no event is enabled. We can, however, model deadlock by defining a null event, e^0, so that $f_{e^0}(\bar{x}) = \bar{x}$.

We associate 'time' indices with the states and events so that $x_k \in \mathscr{X}$ represents the state at time $k \in N$ and $e_k \in g(x_k)$ represents an enabled event at time $k \in N$ (N denotes the set of natural numbers). Notice that there can be just one state at time k, but that many events may be enabled at time k. Should an enabled event e_k occur, then the next state, x_{k+1} is defined by $x_{k+1} = f_{e_k}(x_k)$.

We now define state trajectories and event trajectories. A *state trajectory* is any sequence $\{x_k\} \in \mathscr{X}^N$ such that $x_{k+1} = f_{e_k}(x_k)$ for some $e_k \in g(x_k)$ for all $k \in N$. An *event trajectory* is any sequence $\{e_k\} \in \mathscr{E}^N$ such that there exists a state trajectory, $\{x_k\} \in \mathscr{X}^N$, where for every $k \in N$, $e_k \in g(x_k)$. The set of all such event trajectories is denoted by $\boldsymbol{E} \subset \mathscr{E}^N$. Notice that corresponding to a given event trajectory, there can be only one state trajectory. In general, however, an event trajectory that produces a given state trajectory is not unique. Notice that all state and event trajectories must be of infinite length.

Because not every event trajectory $E \in \boldsymbol{E}$ may be physically realizable, our model allows for a set of valid event trajectories, $\boldsymbol{E}_v \subset \boldsymbol{E}$. $\boldsymbol{E}_v(x_0)$ is the set of valid event trajectories when the initial state is $x_0 \in \mathscr{X}$. The framework provides another mechanism for further pruning $\boldsymbol{E}$. $\boldsymbol{E}_a \subset \boldsymbol{E}_v$ is the set of allowed event trajectories. Including $\boldsymbol{E}_a$ in our model yields a great deal of modelling power. In particular, we will make extensive use of $\boldsymbol{E}_a$ to model the decision-making policies which we impose on our systems.

If we fix $k \in N$, then E_k denotes the sequence of events $e_0, e_1, \ldots, e_{k-1}$, and $E_kE \in \boldsymbol{E}_v(x_0)$ is the concatenation of E_k with the sequence of infinite length $E = e_k, e_{k+1}, \ldots$. The function $X(x_0, E_k, k)$ will be used to denote the state reached at time k from $x_0 \in \mathscr{X}$ by application of event sequence E_k such that $E_kE \in \boldsymbol{E}$. For fixed x_0, the functions $X(x_0, E_k, k)$, where $E_kE \in \boldsymbol{E}_v(x_0)$, are called *motions*.

1.2. *Stability definitions and theorems*

In standard Lyapunov stability theory, we normally speak of stability with respect to one equilibrium point within the state space. However, in the generalized Lyapunov stability theory, we can speak of stability with respect to an *invariant set*. A set is called invariant with respect to G if all motions originating in the set remain in the set. Mathematically, the set $\mathscr{X}_m \subset \mathscr{X}$ is an invariant set with respect to G if $x_0 \in \mathscr{X}_m$ implies that $X(x_0, E_k, k) \in \mathscr{X}_m$ for all $k \in N$ and all E_k such that $E_kE \in \boldsymbol{E}_v(x_0)$.

The machinery that we use to determine the 'distance' between any two states in $\mathscr{X}$ is that of the metric. Let $\rho: \mathscr{X} \times \mathscr{X} \to \mathfrak{R}$ denote a metric on $\mathscr{X}$, and let $\{\mathscr{X}; \rho\}$ denote a metric space. If $\mathscr{X}_z \subset \mathscr{X}$ then the distance between a point $x \in \mathscr{X}$ and the set $\mathscr{X}_z$ is defined by $\rho(x, \mathscr{X}_z) = \inf\{\rho(x, x'); x' \in \mathscr{X}_z\}$. Additionally, we define the r-neighbourhood of a set $\mathscr{X}_z \subset \mathscr{X}$ to be the set $S(\mathscr{X}_z; r) = \{x \in \mathscr{X}: 0 < \rho(x, \mathscr{X}_z) < r\}$ where $r > 0$.

A closed invariant set $\mathscr{X}_m \subset \mathscr{X}$ of G (due to the definition of invariance, all invariant sets are closed with respect to $\{\mathscr{X}; \rho\}$) is called *stable in the sense of Lyapunov* with respect to $\boldsymbol{E}_a$ if for any $\varepsilon > 0$ it is possible to find some $\delta > 0$

such that when $\rho(x_0, \mathscr{X}_m) < \delta$, we have $\rho(X(x_0, E_k, k), \mathscr{X}_m) < \varepsilon$ for all E_k such that $E_k E \in \boldsymbol{E}_a(x_0)$ and $k \in N$. If furthermore $\rho(X(x_0, E_k, k), \mathscr{X}_m) \to 0$ as $k \to \infty$, then the closed invariant set $\mathscr{X}_m$ of G is called asymptotically stable with respect to $\boldsymbol{E}_a$. As is always the case, these properties are local stability properties, i.e. with respect to some r-neighbourhood.

It follows directly from the above definitions of stability that if the closed invariant set $\mathscr{X}_m \subset \mathscr{X}$ is stable (asymptotically stable) in the sense of Lyapunov with respect to $\boldsymbol{E}_a$, then it is stable (asymptotically stable) in the sense of Lyapunov with respect to all $\boldsymbol{E}_{a'}$ such that $\boldsymbol{E}_{a'} \subset \boldsymbol{E}_a$.

If the closed invariant set $\mathscr{X}_m \subset \mathscr{X}$ of G is asymptotically stable with respect to $\boldsymbol{E}_a$, then the set $\mathscr{X}_a \subset \mathscr{X}$ having the property that for all $x_0 \in \mathscr{X}_a$, $\rho(X(x_0, E_k, k), \mathscr{X}_m) \to 0$ for all E_k such that $E_k E \in \boldsymbol{E}_a(x_0)$ as $k \to \infty$ is called the region of asymptotic stability of $\mathscr{X}_m$ with respect to $\boldsymbol{E}_a$. If $\mathscr{X}_a = \mathscr{X}$, then the closed invariant set $\mathscr{X}_m$ of G is called asymptotically stable in the large with respect to $\boldsymbol{E}_a$.

In addition to our concern that eventually $\rho(X(x_0, E_k, k), \mathscr{X}_m) \to 0$, we may be concerned with how quickly any state trajectory must reach the invariant set. In particular, we say that the closed invariant set $\mathscr{X}_m \subset \mathscr{X}$ of G is exponentially stable with respect to $\boldsymbol{E}_a$ if $\rho(X(x_0, E_k, k), \mathscr{X}_m) \leq \zeta e^{-\alpha k} \rho(x_0, \mathscr{X}_m)$ for some $\alpha > 0$ and some $\zeta > 0$ and for all E_k such that $E_k E \in \boldsymbol{E}_a(x_0)$ and $k \in N$.

It follows directly from the definition of exponential stability that if the closed invariant set $\mathscr{X}_m \subset \mathscr{X}$ is exponentially stable with respect to $\boldsymbol{E}_a$, then it is exponentially stable with respect to all $\boldsymbol{E}_{a'}$ such that $\boldsymbol{E}_{a'} \subset \boldsymbol{E}_a$.

If the closed invariant set $\mathscr{X}_m \subset \mathscr{X}$ of G is exponentially stable with respect to $\boldsymbol{E}_a$, then the set $\mathscr{X}_e \subset \mathscr{X}$ having the property that for all $x_0 \in \mathscr{X}_e$, $\rho(X(x_0, E_k, k), \mathscr{X}_m) \leq \zeta e^{-\alpha k} \rho(x_0, \mathscr{X}_m)$ for some $\alpha > 0$ and some $\zeta > 0$ for all E_k and $k \in N$ such that $E_k E \in \boldsymbol{E}_a(x_0)$ is called the *region of exponential stability* of $\mathscr{X}_m$ with respect to $\boldsymbol{E}_a$. If $\mathscr{X}_e = \mathscr{X}$, then the closed invariant set $\mathscr{X}_m$ of G is called *exponentially stable in the large* with respect to $\boldsymbol{E}_a$.

We now state three theorems, whose proofs may be found in Passino *et al.* (1990, 1991), Michel (1992 a, b), which establish necessary and sufficient conditions for a system to possess the stability properties defined above.

Theorem 1: *In order for a closed invariant set $\mathscr{X}_m \subset \mathscr{X}$ of G to be stable in the sense of Lyapunov with respect to $\boldsymbol{E}_a$, it is necessary and sufficient that in a sufficiently small neighborhood $S(\mathscr{X}_m; r)$ of the set $\mathscr{X}_m$ there exists a specified functional V with the following properties.*

(i) *For all sufficiently small $c_1 > 0$, it is possible to find a $c_2 > 0$ such that $V(x) > c_2$ for $x \in S(\mathscr{X}_m; r)$ and $\rho(x, \mathscr{X}_m) > c_1$.*

(ii) *For any $c_4 > 0$ as small as desired, it is possible to find a $c_3 > 0$ so small that when $\rho(x, \mathscr{X}_m) < c_3$ for $x \in S(\mathscr{X}_m; r)$ we have $V(x) \leq c_4$.*

(iii) *$V(X(x_0, E_k, k))$ is a non-increasing function for $x_0 \in S(\mathscr{X}_m; r)$ and for all $k \in N$, provided that $X(x_0, E_k, k) \in S(\mathscr{X}_m; r)$ for all E_k such that $E_k E \in \boldsymbol{E}_a(x_0)$.*

Theorem 2: *In order for a closed invariant set $\mathscr{X}_m \subset \mathscr{X}$ of G to be asymptotically stable in the sense of Lyapunov with respect to $\boldsymbol{E}_a$, it is necessary and sufficient that in a sufficiently small neighbourhood, $S(\mathscr{X}_m; r)$, of the set $\mathscr{X}_m$ there exists a specified functional V having properties* (i), (ii), *and* (iii) *of Theorem 1* *and furthermore* $V(X(x_0, E_k, k)) \to 0$ *as* $k \to \infty$ *for all* E_k *such that* $E_k E \in \boldsymbol{E}_a(x_0)$ *and for all* $k \in N$ *as long as* $X(x_0, E_k, k) \in S(\mathscr{X}_m; r)$.

Theorem 3: *The closed invariant set* $\mathscr{X}_m \subset \mathscr{X}$ *of* G *is exponentially stable with respect to* $\boldsymbol{E}_a$ *if there exists a functional* V *defined on* $S(\mathscr{X}_m; r)$, $D \in \{1, 2, \ldots\}$ *and* $c_1, c_2, c_3 > 0$ *with* $c_3/c_2 \in (0, 1)$ *such that*

(i) $c_1 \rho(x, \mathscr{X}_m) \leq V(x) \leq c_2 \rho(x, \mathscr{X}_m)$ *for all* $x \in S(\mathscr{X}_m; r)$;

(ii) $V(X(x_0, E_{k+D}, k + D)) - V(X(x_0, E_k, k)) \leq -c_3 \rho(X(x_0, E_k, k), \mathscr{X}_m)$ *for* $x_0 \in S(\mathscr{X}_m; r)$ *and for all* $k \in N$, *provided that* $X(x_0, E_k, k) \in S(\mathscr{X}_m; r)$ *for all* E_k *such that* $E_k E \in \boldsymbol{E}_a(x_0)$.

The conditions of Theorems 2 and 3 are sufficient for asymptotic stability in the large and exponential stability in the large, respectively, if they are changed so that all occurrences of $S(\mathscr{X}_m; r)$ are replaced by $\mathscr{X}$.

A conventional Lyapunov approach to stability analysis will be taken where we define ρ and the invariant set $\mathscr{X}_m$, choose a Lyapunov function, V, and show that it satisfies the appropriate conditions of the above theorems so that we can infer that the system possesses certain stability properties.

1.3. *Summary*

Above, we have indicated the types of load balancing problems to be considered and have established a modelling formalism for load balancing systems. In addition, we have provided stability definitions and an approach to stability analysis for load balancing systems. In § 2 we will study the load balancing problem without delays in passing and sensing load. We prove in this case that a particular load redistribution policy is asymptotically and exponentially stable. We also generalize on the original non-delay load balancing system and prove that several of the generalized systems are asymptotically and exponentially stable. The generalizations include generalized load passing conditions, virtual load systems, and discrete load systems. In § 3, we will study the load balancing problem with delays in passing and sensing load. We prove in this case that a particular load redistribution policy is asymptotically and exponentially stable. Finally, we offer some concluding remarks in § 4.

2. A load balancing problem without delays

The load processors, $L = \{1, 2, \ldots, N\}$, are all connected to a network along which they can pass load to other load processors. The network of load processors is described by a directed graph, (L, A), where $A \subset L \times L$. For every $i \in L$, there must exist $(i, j) \in A$ in order to assure that every load processor is connected to the network, and if $(i, j) \in A$ then $(j, i) \in A$. Load processor i can only transfer a portion of its load to load processor j if $(i, j) \in A$. Finally, if $(i, j) \in A$, then $i \neq j$.

Each load processor $i \in L$ has a buffer in which its load is stored prior to processing. It is the buffer levels x_i that we actually wish to balance; thus, it is the buffer levels that are affected by load transfers. In this section, we will assume that the load can be partitioned into sufficiently small units so that it is valid to describe it with a continuous variable. We will also assume that the total amount of load in the buffers of the load processors on the network remains static until a load balance is achieved; hence, we assume that no load arrives or is processed during the balancing of the load.

In this section, we are not considering load transportation or load sensing delays. Hence, we require that the real time between events e_k and e_{k+1} (which will represent the passing of load) is greater than the greatest system transportation time plus the greatest system sensing time. We do, however, allow for more than one node to pass load at one time and for nodes simultaneously to pass load to more than one of the nodes that they are connected to on the network.

Let $\mathscr{X} = \Re^N$ be the set of states and $\boldsymbol{x}_k = [x_1 x_2 \dots x_N]^t$ and $\boldsymbol{x}_{k+1} = [x'_1 x'_2 \dots x'_N]^t$ denote the states at times k and $k+1$, respectively. Let $x_i(k')$ denote the amount of load at node $i \in L$ at time k'. Let $e^{i,p(i)}_{\alpha(i)}$ represent that node $i \in L$ passes load to its neighbours $m \in p(i)$ where $p(i) = \{j\colon (i, j) \in A\}$. Let the list $\alpha(i) = (\alpha_j(i), \alpha_{j'}(i), \dots, \alpha_{j''}(i))$ such that $j < j' < \cdots < j''$ and $j, j', \dots, j'' \in p(i)$ and $\alpha_j \geqslant 0$ for all $j \in p(i)$; the size of the list $\alpha(i)$ is $|p(i)|$. For convenience, we will denote this list by $\alpha(i) = (\alpha_j(i)\colon j \in p(i))$. $\alpha_{\mathrm{m}}(i)$ denotes the amount of load transferred from $i \in L$ to $m \in p(i)$. Let $\{e^{i,p(i)}_{\alpha(i)}\}$ denote the set of all possible such load transfers. Let the set of events be described by

$$\mathscr{E} = \mathscr{P}(\{e^{i,p(i)}_{\alpha(i)}\}) - \{\phi\}$$

where $\mathscr{P}(Q)$ denotes the power set of the set Q. Notice that each event $e_k \in \mathscr{E}$ is defined as a set, with each element of e_k representing the passing of load by some node $i \in L$ to its neighbouring nodes in the network. Let $\gamma_{ij} \in (0, 1)$ for $(i, j) \in A$ represent the proportion of the load imbalance that is *sometimes* guaranteed to be reduced when i passes load to j.

Below, we specify g and f_{e_k} for $e_k \in g(\boldsymbol{x}_k)$.

(1) Event $e_k \in g(\boldsymbol{x}_k)$ if both (a) and (b) below hold.

(a) For all $e^{i,p(i)}_{\alpha(i)} \in e_k$ where $\alpha(i) = (\alpha_j(i)\colon j \in p(i))$ it is the case that:

(i) $\alpha_j(i) = 0$ if $x_i \leqslant x_j$ where $j \in p(i)$;

(ii) $0 \leqslant \sum_{m \in p(i)} \alpha_m(i) \leqslant x_i - (x_j + \alpha_j(i))$ for all $j \in p(i)$ such that $x_i > x_j$; and

(iii) $\alpha_{j^*}(i) \geqslant \gamma_{ij^*}(x_i - x_{j^*})$ for some $j^* \in \{j\colon x_j \leqslant x_m$ for all $m \in p(i)\}$.

Condition (i) prevents load from being passed by node i to node j if node i is less heavily loaded than node j. Condition (ii) directly implies that $x_i - \sum_{m \in p(i)} \alpha_m(i) \geqslant x_j + \alpha_j(i)$. Thus, after the load $\alpha(i)$ has been passed, the remaining load of node i must be at least as large as $x_j + \alpha_j(i)$ for every node $j \in p(i)$ that was less heavily loaded than node i to begin with. Condition (iii) implies that if node i is not load balanced with all of its neighbours and it passes load, then i must pass a non-negligible portion of its load to some least-loaded neighbour j^*.

(b) If $e^{i,p(i)}_{\alpha(i)} \in e_k$ where $\alpha(i) = (\alpha_j(i)\colon j \in p(i))$, then $e^{i,p(i)}_{\delta(i)} \notin e_k$ where $\delta(i) = (\delta_j(i)\colon j \in p(i))$ if $\alpha_j(i) \neq \delta_j(i)$ for some $j \in p(i)$. Hence, in each valid event e_k, there must be a consistent definition of the load to be passed from any node i to any node j, $\alpha_j(i)$.

(2) If $e_k \in g(\boldsymbol{x}_k)$ and $e^{i,p(i)}_{\alpha(i)} \in e_k$ then $f_{e_k}(\boldsymbol{x}_k) = \boldsymbol{x}_{k+1}$ where

$$x'_i = x_i - \sum_{\{j: j \in p(i)\}} \alpha_j(i) + \sum_{\{j: i \in p(j), e^{j,p(j)}_{\alpha(j)} \in e_k\}} \alpha_i(j).$$

The load of node i at time $k+1$, x'_i, is the load of node i at time k minus the total load passed by node i at time k plus the total load received by node i at time k.

Let $\boldsymbol{E}_v = \boldsymbol{E}$ be the set of valid event trajectories. We must further specify the sets of allowed event trajectories. Define a *partial event of type* i to represent the passing of $\alpha(i)$ amount of load from $i \in L$ to its neighbours $p(i)$. A partial event of type i will be denoted by $e^{i,p(i)}$ and the occurrence of $e^{i,p(i)}$ indicates that $i \in L$ attempts to balance its load further with its neighbours. Event $e_k \in g(\boldsymbol{x}_k)$ is composed of a set of partial events. Next, we define two possibilities for the allowed event trajectories $\boldsymbol{E}_a$.

(i) For $\boldsymbol{E}_i \subset \boldsymbol{E}_v$, assume that each type of partial event occurs infinitely often on each $E \in \boldsymbol{E}_i$.

(ii) For $\boldsymbol{E}_B \subset \boldsymbol{E}_v$, assume that there exists $B > 0$ such that for every event trajectory $E \in \boldsymbol{E}_B$, in every substring $e_{k'}, e_{k'+1}, e_{k'+2}, \ldots, e_{k'+(B-1)}$ of E there is the occurrence of every type of partial event (i.e. for every $i \in L$ partial event $e^{i,p(i)} \in e_k$ for some k, $k' \leqslant k \leqslant k' + B - 1$).

Clearly

$$\mathscr{X}_b = \{x_k \in \mathscr{X}: \boldsymbol{x}_i = \boldsymbol{x}_j \text{ for all } (i, j) \in A\}$$

is an invariant set that represents a perfectly balanced load. Notice that the only $e_k \in g(\boldsymbol{x}_k)$, when $\boldsymbol{x}_k \in \mathscr{X}_b$, are such that all $e^{i,p(i)}_{\alpha(i)} \in e_k$ have $\alpha(i) = (0, 0, \ldots, 0)$.

If $\boldsymbol{E}_a = \boldsymbol{E}_B \subset \boldsymbol{E}_i$, the load balancing problem described above is the same as the one of Tsitsiklis and Bertsekas (1989), except that in this section we do not allow delays in transporting and sensing load. In § 4 we will study load balancing systems with delays.

2.1. *Asymptotic convergence to a balanced state*

To study the ability of the system to redistribute load automatically to achieve balancing, we use a Lyapunov stability theoretic approach. Let $\bar{\boldsymbol{x}} = [\bar{x}_1 \ \ldots \ \bar{x}_N]$. Choose

$$\rho(\boldsymbol{x}_k, \mathscr{X}_b) = \inf \{\max \{|x_1 - \bar{x}_1|, \ldots, |x_N - \bar{x}_N|\}: \bar{\boldsymbol{x}} \in \mathscr{X}_b\} \tag{1}$$

The following result provides slightly weaker conditions for load balancing than in Tsitsiklis and Bertsekas (1989) and sets the stage for studying exponential stability in the next subsection and generalizing the load balancing results of Tsitsiklis and Bertsekas (1989) in § 3.

Theorem 4: *For the load processor network system described above, the invariant set $\mathscr{X}_b$ is asymptotically stable in the large with respect to $\boldsymbol{E}_i$.*

Proof: Choose

$$V(\boldsymbol{x}_k) = \max_i \left\{ \frac{1}{N} \sum_{j=1}^{N} x_j - x_i \right\} \tag{2}$$

Notice that

$$\frac{1}{N}\sum_{j=1}^{N} x_j \geq \frac{1}{N}[\max_i \{x_i\} + (N-1)\min_i \{x_i\}] \tag{3}$$

It is clear from (1), (2) and (3) that the following relations are valid.

$$\rho(\boldsymbol{x}_k, \mathcal{X}_b) \geq \frac{1}{2}(\max_i \{x_i\} - \min_i \{x_i\}) \tag{4}$$

$$\rho(\boldsymbol{x}_k, \mathcal{X}_b) \leq \max_i \{x_i\} - \min_i \{x_i\} \tag{5}$$

$$V(\boldsymbol{x}_k) = \frac{1}{N}\sum_{j=1}^{N} x_j - \min_i \{x_i\} \leq \max_i \{x_i\} - \min_i \{x_i\} \tag{6}$$

$$V(\boldsymbol{x}_k) \geq \frac{1}{N}[\max_i \{x_i\} + (N-1)\min_i \{x_i\}] - \min_i \{x_i\} \tag{7}$$

Equations (4) and (6) yield $2\rho(\boldsymbol{x}_k, \mathcal{X}_b) \geq \max_i \{x_i\} - \min_i \{x_i\} \geq V(\boldsymbol{x}_k)$, so that condition (ii) of Theorem 1 is satisfied. Equation (7) can be manipulated to yield

$$V(\boldsymbol{x}_k) \geq \frac{1}{N}\left(\max_i \{x_i\} - \min_i \{x_i\}\right) \tag{8}$$

Equations (5) and (8) directly imply that $V(\boldsymbol{x}_k) \geq (1/N)\rho(\boldsymbol{x}_k, \mathcal{X}_b)$, so that condition (i) of Theorem 1 is satisfied.

To satisfy the final condition of Theorem 1, we must show that $V(X(\boldsymbol{x}_0, E_k, k))$ is a non-increasing function for all $k \in N$, all $\boldsymbol{x}_0 \in S(\mathcal{X}_b; r)$ and all E_k such that $E_k E \in \boldsymbol{E}_i(\boldsymbol{x}_0)$. To see that this is the case, notice that once $\boldsymbol{x}_0$ is specified, $V(\boldsymbol{x}_k)$ varies only as the lightest load in the network, $\min_i \{x_i\} = x_{j^{**}}$, varies. The most lightly loaded node in the network cannot possibly pass load, so $x'_{j^{**}} \geq x_{j^{**}}$. Assume an event $e_k \in g(\boldsymbol{x}_k)$ occurs. According to condition (ii) on $e_k \in g(\boldsymbol{x}_k)$, if $e^{i,p(i)}_{\alpha(i)} \in e_k$ and $j^{**} \in p(i)$, it is not possible that $x'_i < x_{j^{**}} + \alpha_{j^{**}}(i)$. Therefore, $\min_i \{x'_i\} \geq x_{j^{**}}$ and $V(\boldsymbol{x}_{k+1}) \leq V(\boldsymbol{x}_k)$. Thus, condition (iii) of Theorem 1 is satisfied and $\mathcal{X}_b$ is stable in the sense of Lyapunov with respect to $\boldsymbol{E}_i$.

In order to show that $\mathcal{X}_b$ is asymptotically stable in the large with respect to $\boldsymbol{E}_i$, we must show that for all $\boldsymbol{x}_0 \notin \mathcal{X}_b$ and all E_k such that $E_k E \in \boldsymbol{E}_i(\boldsymbol{x}_0)$, $V(X(\boldsymbol{x}_0, E_k, k)) \to 0$ as $k \to \infty$. If $\boldsymbol{x}_k \notin \mathcal{X}_b$, then there must be some lightest loaded node j^{**} (there may be more than one such node) and some other node i such that $(i, j^{**}) \in A$ and $x_i > x_{j^{**}}$. Because of the restrictions imposed by $\boldsymbol{E}_i$, we know that all the partial events are guaranteed to occur infinitely often. According to condition (a)(iii) on $e_k \in g(\boldsymbol{x}_k)$, each time partial event $e^{i,p(i)}$ occurs, $x_{j^{**}}$ is guaranteed to increase by a fixed fraction $\gamma_{ij^{**}} \in (0, 1)$ of $x_i - x_{j^{**}}$ so that $x'_{j^{**}} > x_{j^{**}}$. Thus, regardless of how many lightest loaded nodes there are, it is inevitable that eventually the overall lightest load in the network must increase. Hence, for every $k \geq 0$, there exists $k' \geq k$ such that $V(\boldsymbol{x}_{k'}) > V(\boldsymbol{x}_{k'+1})$ as long as $\boldsymbol{x}_{k'} \notin \mathcal{X}_b$ so that $V(X(\boldsymbol{x}_0, E_k, k)) \to 0$ as $k \to \infty$ and $\mathcal{X}_b$ is asymptotically stable in the large with respect to $\boldsymbol{E}_i$. □

Remark 1: Notice that we do not need the restrictions on allowed event trajectories that are imposed by $\boldsymbol{E}_i$ to support our conclusion of stability in the sense of Lyapunov. Hence, $\mathscr{X}_b$ is stable in the sense of Lyapunov with respect to $\boldsymbol{E}_v$ as well. □

Remark 2: Note that $\mathscr{X}_b$ is not asymptotically stable in the large with respect to $\boldsymbol{E}_v$. This is due to the fact that without the restrictions on $\boldsymbol{E}_v$ to obtain $\boldsymbol{E}_i$, it is possible that only one $i \in L$ attempts to balance its load for all time. □

Remark 3: Notice that condition (a)(i) on $e_k \in g(\boldsymbol{x}_k)$ is absolutely necessary. If condition (i) is removed, then it is possible that nodes may pass load to their more heavily loaded neighbours. In this case, node j^{**} (where $x_{j^{**}} = \min_m \{x_m: m \in L\}$) may pass load and $x'_{j^{**}} < x_{j^{**}}$. Hence, the lightest load in the network may decrease and both the proof of Lyapunov stability and the proof of asymptotic stability become invalid. □

Remark 4: Consider the implications of replacing condition (a)(ii) on $e_k \in g(\boldsymbol{x}_k)$ with the more liberal condition

$$0 \leq \sum_{j \in p(i)} \alpha_j(i) \leq x_i - x_j \text{ for all } j \in p(i) \text{ such that } x_i \geq x_j.$$

This new condition implies that if $e^{i,p(i)}_{\alpha(i)} \in e_k$, $\alpha_{j^*}(i)$ (where $j^* \in \{j: x_j \leq x_m$ for all $m \in p(i)\}$) may be such that $x'_i = x_{j^*}$ and $x'_{j^*} = x_i$. In this case, nodes i and j^* simply exchange load levels. It is still true that the lightest load in the network cannot decrease, however, it is not necessarily true that the lightest load in the network will ever increase. Hence, $\mathscr{X}_b$ remains stable in the sense of Lyapunov with respect to $\boldsymbol{E}_i$, but we can no longer claim that $\mathscr{X}_b$ is asymptotically stable with respect to $\boldsymbol{E}_i$. □

Remark 5: Consider eliminating condition (a)(iii) on $e_k \in g(\boldsymbol{x}_k)$. In this case, if $e^{i,p(i)}_{\alpha(i)} \in e_k$ and $j^{**} \in p(i)$ (where $x_{j^{**}} = \min_m \{x_m: m \in L\}$), it is no longer true that $x_{j^{**}}$ must increase by a fixed fraction of $x_i - x_{j^{**}}$. It is now possible that even if $e^{i,p(i)}_{\alpha(i)} \in e_k$ for all $k > k'$, $x_{j^{**}} \not\to x_i$ as $k \to \infty$. For example, $x_i - x_{j^{**}}$ may be reduced after each load passing by factors of $1/(k+1)^2$ and the two loads will never converge to each other. Hence, it is no longer true that $\mathscr{X}_b$ is asymptotically stable with respect to $\boldsymbol{E}_i$, but it is still the case that $\mathscr{X}_b$ is stable in the sense of Lyapunov with respect to $\boldsymbol{E}_i$. □

These remarks are similar in nature to the questions posed by Tsitsiklis and Bertsekas (1989) after their discussion of the load balancing problem.

2.2. *Exponential convergence to a balanced state*

We now say something about the rate at which the system converges to a balanced state. In order to do this, we employ Theorem 3. If we satisfy the conditions of this theorem, we know that $\rho(\boldsymbol{x}_k, \mathscr{X}_b)$ will be bounded from above by an exponential $\zeta e^{-\alpha k}\rho(\boldsymbol{x}_0, \mathscr{X}_b)$ for some $\alpha > 0$ and $\zeta > 0$.

Theorem 5: *For the load processor network system described above, the invariant set $\mathscr{X}_b$ is exponentially stable in the large with respect to $\boldsymbol{E}_B$.*

Proof: For the proof, see Appendix A. □

Remark 1: The proof of Theorem 5 depends critically upon the fact that $\boldsymbol{E}_B$ requires that for every $i \in L$, the corresponding partial event, $e^{i,p(i)}$, occurs at least once in every B events. Hence, it is clear that $\mathscr{X}_b$ is not exponentially stable in the large with respect to $\boldsymbol{E}_i$. □

Remark 2: In the proof in Appendix A, it is shown that

$$V(\boldsymbol{x}_k) - V(\boldsymbol{x}_{k+N^2B}) \geqslant \gamma^{N^3B}\rho(\boldsymbol{x}_k, \mathscr{X}_b) \tag{9}$$

where

$$V(\boldsymbol{x}_k) = \max_i \left\{ \frac{1}{N}\sum_{j=1} N x_j - x_i \right\}$$

The constant γ^{N^3B} from (9) is directly related to the α from the exponential overbounding function $\zeta e^{-\alpha k}\rho(\boldsymbol{x}_0, \mathscr{X}_b)$. Thus, if the speed of convergence is a design factor, then γ should be made as large as possible and N and B should be made as small as possible.

It is evident that (9) is unnecessarily conservative. Equation (16) from the proof in Appendix A, restated here

$$x_j(k') \geqslant \min_i \{x_i\} + \gamma^{k'-k}[x_i - \min_i \{x_i\}] \text{ for all } k' \geqslant k + NB,\ j \in p(i)$$

is also unnecessarily conservative. Actually, equation (16) is valid for all $k' \geqslant k + RB$, where $R = \max_i \{|p(i)|\}$. Let S be the maximum number of arcs that must be spanned to reach any node $j \in L$ from any other node $i \in L$. N can be replaced by S in (24), restated here

$$x_j(k') \geqslant \min_i \{x_i\} + (\gamma^{k'-k})^N[x_i - \min_i \{x_i\}]$$

and (9) becomes

$$V(\boldsymbol{x}_k) - V(\boldsymbol{x}_{k+RSB}) \geqslant \gamma^{RS^2B}\rho(\boldsymbol{x}_k, \mathscr{X}_b)$$

Therefore, convergence can be accelerated by designing for RS^2 as small as possible. □

Consider three common network topologies of N nodes. If N nodes are connected in a line (e.g. see Fig. 1), then $R = 2$ and $S = N - 1$. If N nodes are connected in a simple ring, then $R = 2$ and $S = \text{int}(N/2)$ ($\text{int}(x)$ is the integer portion of x). If N nodes are completely connected (each node is connected to every other node), then $R = N - 1$ and $S = 1$. In general, the ring network will converge more rapidly than the line network, and the completely connected network will converge more quickly than the ring network. Intuitively, this is what we would expect; convergence performance seems directly related to $|A|$.

Remark 3: If we change our assumptions regarding the network topology to allow networks that are strongly connected, the above analysis may be simply amended to remain valid. We must replace N in (24) with S, where S is defined as in Remark 2. Equation (9) must then be changed by replacing N^3 with S^2N. If $S > N$, then the guaranteed rate of convergence for a strongly connected network with N nodes is slower than for a network with N nodes that satisfies our original network topology assumption. However, if the cost of inter-node connections is great, the sacrifice in convergence speed may be worthwhile. $|A|$ for a strongly connected network of N nodes has a minimum value of N when the nodes are joined in a ring such that if $(i, j) \in A$ then $(j, i) \notin A$.

3. Generalizations of the load balancing problem

In this section we discuss generalizations of the load balancing problem previously outlined. First, we discuss less restrictive conditions on the amount of load that can be passed from node to node, coupled with a new specification of $\boldsymbol{E}_{\mathrm{a}}$. Secondly, we discuss the idea of virtual load, a mechanism to account for the varied rates at which inter-network processors may process load. Finally, we consider the case in which the load in the network cannot be accurately modelled by a continuous variable (i.e. the discrete load case).

3.1. *Generalized load passing conditions*

We will require that condition (i) on $e_k \in g(x_k)$ remain unchanged. We change condition (ii), however, to allow that if $e_{\alpha(i)}^{i,p(i)} \in e_k$ then possibly after the passing of $\alpha(i)$, the load of node i can fall to the level of some node $j' \in p(i)$. This new condition (ii) is

$$\text{(ii}\,a) \quad 0 \leqslant \sum_{m \in p(i)} \alpha_m(i) \leqslant x_i - (x_{j'} + \alpha_{j'}(i)) \text{ for some}$$

$$j' \in \{j\colon x_j < x_i,\ j \in p(i)\}$$

Condition (iii) is also changed because we no longer require that if $e_{\alpha(i)}^{i,p(i)} \in e_k$ then node i passes a non-negligible amount of load to some least loaded neighbour j^*. The γ_{ij} are fixed *a priori* as before. The new condition (iii) is

$$\text{(iii}\,a) \quad \alpha_{j'}(i) \geqslant \gamma_{ij'}(x_i - x_j') \text{ for some } j' \in p(i) \text{ such that } x_{j'} < x_i$$

Notice that we now require only that if $e_{\alpha(i)}^{i,p(i)} \in e_k$, then node i passes a predefined fraction of the load difference between nodes i and j' for *some* node $j' \in p(i)$.

We now define new sets of allowed event trajectories. We define an *elementary event*, $e_{\alpha_j(i)}^{ij}$ to represent the passing of load $\alpha_j(i)$ from processor i to processor j (note that $e_{\alpha(i)}^{i,p(i)} = \{e_{\alpha_j(i)}^{ij}\colon j \in p(i)\}$). We define an elementary event of type (i, j) to be any $e_{\alpha_j(i)}^{ij}$, and denote an elementary event of type (i, j) with e^{ij}.

(i) For $\boldsymbol{E}_{\mathrm{I}} \subset \boldsymbol{E}_{\mathrm{v}}$, every event trajectory $E \in \boldsymbol{E}_{\mathrm{I}}$ must contain an infinite number of occurrences of elementary events of every type e^{ij} for all $(i, j) \in A$.

(ii) For $\boldsymbol{E}_{B'} \subset \boldsymbol{E}_{\mathrm{v}}$, assume that there exists $B' > 0$ such that for every event trajectory $E \in \boldsymbol{E}_{B'}$, in every substring $e_{k'}, e_{k'+1}, e_{k'+2}, \ldots, e_{k'+(B'-1)}$ of E there is the occurrence of every type of elementary event (i.e. for every $i \in L$ elementary event $e^{ij} \in e_k$ for some k, $k' \leqslant k \leqslant k' + B' - 1$).

Theorem 6: *For the load processor network system with conditions* (ii a) *and* (iii a) *the invariant set* $\mathscr{X}_{\mathrm{b}}$ *is asymptotically stable in the large with respect to* $\boldsymbol{E}_{\mathrm{I}}$.

Proof: Using the same ρ and V as in Theorem 4, the proof for stability in the sense of Lyapunov with respect to $\boldsymbol{E}_{\mathrm{I}}$ is the same as in the proof of Theorem 4. The proof of asymptotic stability in the large, however, must be slightly modified. In the original proof, we are guaranteed that the partial event $e^{i,p(i)}$, where $(i, j^{**}) \in A$ and $x_i > x_{j^{**}}$, must occur infinitely often. Given the above generalizations, we can simply state that the elementary event $e^{ij^{**}}$, where

$x_i > x_{j^{**}}$, must occur infinitely often or until $\boldsymbol{x}_k \in \mathscr{X}_b$. Thus, we can say that the overall lightest load in the network must definitely increase an infinite number of times or until it is equal to the average network load. Hence, we have that for the generalized load system, $\mathscr{X}_b$ is asymptotically stable in the large with respect to $\boldsymbol{E}_I$. □

Remark 1: The new conditions allow for greater efficiency because it is no longer necessary for node i to examine all x_j with $j \in p(i)$ to find x_{j^*} before passing. In a network where $|p(i)|$ is large, this may prove to be quite a time-saving advantage. □

Theorem 7: *For the load processor network system with conditions* (ii *a*) *and* (iii *a*) *the invariant set* $\mathscr{X}_b$ *is exponentially stable in the large with respect to* $\boldsymbol{E}_{B'}$.

The proof is omitted as it is very similar to the proof of Theorem 5.

3.2. *Virtual load*

In practice, it is often the case that the load processors in the network may process the load at different rates. In this case, it is useful to scale the physical load of each processor by assigning constants $\beta_i > 0$, which are inversely proportional to the rate at which processor i can process load, for each $i \in L$. Hence, we define $\beta_i x_i$ as the *virtual* load of processor i, and it is the virtual load that we wish to balance among the network nodes. It is useful to balance the virtual load in a load processor network to ensure that nodes which process load faster have a larger portion of the available load.

With a few adjustments, the above analysis applies directly in the case of virtual load. First of all, because we are interested in balancing the virtual load, we should only allow node i to pass load to node j if the virtual load of node i is greater than the virtual load of node j. Accordingly, condition (i) on $e_k \in g(\boldsymbol{x}_k)$ must be changed to

$$\text{(i}\,b) \quad \alpha_j = 0 \text{ if } \beta_i x_i \leqslant \beta_j x_j \text{ where } j \in p(i)$$

Secondly, we require that after node i passes load, its virtual load be at least as large as the possibly increased virtual load, due to $\alpha_{j^*}(i)$, of node j^*. This requirement can be expressed as

$$\beta_i\left(x_i - \sum_{j \in p(i)} \alpha_j(i)\right) \geqslant \beta_{j^*}(x_{j^*} + \alpha_{j^*}(i))$$

Direct manipulation of this equation leads to the extension of condition (ii *a*)

$$\text{(ii}\,b) \quad 0 \leqslant \sum_{j \in p(i)} \alpha_j(i) \leqslant x_i - \frac{\beta_{j^*}}{\beta_i}(x_{j^*} + a_{j^*}(i)) \text{ for all}$$
$$j^* \in \{j\colon \beta_j x_j \leqslant \beta_m x_m \text{ for all } m \in p(i)\}$$

We also must require that if node i is not virtual load balanced with all of its neighbours, then i must pass a non-negligible portion of its load to at least one of its neighbours. We can express this condition as

$$\tfrac{1}{2}\{(\beta_i x_i - \beta_j x_j) - [\beta_i(x_i - \alpha_j(i)) - \beta_j(x_j + \alpha_j(i))]\} \geqslant \gamma_{ij}(\beta_i x_i - \beta_j x_j)$$

for some $j \in p(i)$. After some manipulation, we arrive at the virtual load version of condition (iii *a*)

$$\text{(iii } b) \quad \alpha_j(i) \geq \frac{2\gamma_{ij}(\beta_i x_i - \beta_j x_j)}{\beta_i + \beta_j} \text{ for some } j \in p(i)$$

Notice that in the case of $\beta_i = 1$ for all $i \in L$, the conditions (i *b*), (ii *b*) and (iii *b*) properly reduce to conditions (i), (ii *a*), and (iii *a*).

Clearly

$$\mathscr{X}_{\mathrm{bv}} = \{\boldsymbol{x}_k \in \mathscr{X}: |\beta_i x_i - \beta_j x_j| = 0 \text{ for all } (i, j) \in A\}$$

is an invariant set which represents a perfectly balanced virtual load.

Theorem 8: *For the virtual load processor network system with conditions* (i *b*), (ii *b*), *and* (iii *b*) *the invariant set* $\mathscr{X}_{\mathrm{bv}}$ *is exponentially stable in the large with respect to* $\boldsymbol{E}_{B'}$.

If all references to x_i are replaced by references to $\beta_i x_i$ and the new conditions on $e_k \in g(\boldsymbol{x}_k)$ are observed, the proof is very similar to the proof of Theorem 5.

Remark 1: In the virtual load balancing problem, it is of course necessary that node i not only has knowledge of x_j for all $j \in p(i)$, but also of β_j for all $j \in p(i)$. □

Remark 2: Just as new load can enter the load balanced system, perturbing the balance, the load processing capabilities of the load processors may change, perturbing the balance of the virtual load balanced system. Given that the $\beta = \{\beta_i: i \in L\}$ is updated to reflect the change in load processing capability (e.g. a change in the rate at which some node can process load), the system will recognize the imbalance and begin to rebalance from a new state $\boldsymbol{x}_0 \notin \mathscr{X}_{\mathrm{bv}}$. □

3.3. *Discrete load*

Consider now that we have the same system as originally described, except that in this case, we may not assume that the load can be described with a continuous variable, as is the case in many practical systems. In fact, we assume that the load in the system is partitioned into blocks. The largest block in the network has size $M > 0$ and the smallest block in the network has size m, $M \geq m > 0$. In contrast to the perfect load balancing that is possible in the continuous load case, the best we can generally hope to do with only local information in the discrete load case is to balance each interprocessor connection to within M. Next, we define the model G for the discrete load case.

We utilize the same $\mathscr{X}$ and $\mathscr{E}$ as in the continuous load case. Below, we specify g and f_e for $e_k \in g(\boldsymbol{x}_k)$.

(1) Event $e_k \in g(\boldsymbol{x}_k)$ if both (*a*) and (*b*) below hold.

(*a*) For all $e_{\alpha(i)}^{i,p(i)} \in e_k$ where $\alpha(i) = (\alpha_j(i): j \in p(i))$ it is the case that:

(i) $\alpha_j(i) = 0$ if $x_i - x_j \leq M$ where $j \in p(i)$.

(ii) $x_i - \sum_{m \in p(i)} \alpha_m(i) > \min_j \{x_j: j \in p(i)\}$.

(iii) If $\alpha_j(i) > 0$ for some $j \in p(i)$, then
$\alpha_{j^*}(i) > 0$ for some $j^* \in \{j: x_j \leq x_m \text{ for all } m \in p(i)\}$.

Condition (i) prevents load from being passed by node i to node j if nodes i and j are balanced within M. Condition (ii) implies that after the load $\alpha(i)$ has been passed, the remaining load of node i must be larger than the load at time k of some neighbour of i. Condition (iii) implies that if node i is not load balanced to within M with all of its neighbours, then i must pass some load to some least-loaded neighbour j^*.

(*b*) If $e_{\alpha(i)}^{i,p(i)} \in e_k$ where $\alpha(i) = (\alpha_j(i)\colon\ j \in p(i))$, then $e_{\delta(i)}^{i,p(i)} \notin e_k$ where $\delta(i) = \{\delta_j(i)\colon\ j \in p(i)\}$ if $\alpha_j(i) \neq \delta_j(i)$ for some $j \in p(i)$. Hence, in each valid event e_k, there must be a consistent definition of the load to be passed from any node i to any other node j, $\alpha_j(i)$.

(2) If $e_k \in g(\boldsymbol{x}_k)$ and $e_{\alpha(i)}^{i,p(i)} \in e_k$ then $f_{e_k}(\boldsymbol{x}_k) = \boldsymbol{x}_{k+1}$ where

$$x_i' = x_i - \sum_{\{j:j \in p(i)\}} \alpha_j(i) + \sum_{\{j:i \in p(j), e_{\alpha(j)}^{j,p(j)} \in e_k\}} \alpha_i(j)$$

The load of node i at time $k+1$, x_i', is the load of node i at time k minus the total load passed by node i at time k plus the total load received by node i at time k.

Let $\boldsymbol{E}_{\text{v}} = \boldsymbol{E}$ be the set of valid event trajectories. Define $\boldsymbol{E}_i$, $\boldsymbol{E}_B \subset \boldsymbol{E}_{\text{v}}$ as in the continuous load case.

Clearly

$$\mathscr{X}_{\text{bd}} = \{\boldsymbol{x}_k \in \mathscr{X}\colon |x_i - x_j| \leq M \text{ for all } (i, j) \in A\}$$

is an invariant set that represents a balanced load in the sense described above. Notice that the only $e_k \in g(\boldsymbol{x}_k)$, where $\boldsymbol{x}_k \in \mathscr{X}_{\text{bd}}$, are ones such that all $e_{\alpha(i)}^{i,p(i)} \in e_k$ have $\alpha(i) = (0, 0, \ldots, 0)$.

Once again, we employ a Lyapunov stability theoretic approach. Let $\bar{\boldsymbol{x}} = [\bar{x}_1, \ldots, \bar{x}_N]$. Choose

$$\rho(\boldsymbol{x}_k, \mathscr{X}_{\text{bd}}) = \inf\{\max\{|x_i - \bar{x}_i|\colon i \in L\}\colon \bar{\boldsymbol{x}} \in \mathscr{X}_{\text{bd}}\} \tag{10}$$

Theorem 9: *For the discrete load processor network system, the invariant set* $\mathscr{X}_{\text{bd}}$ *is asymptotically stable in the large with respect to* $\boldsymbol{E}_i$.

Proof: For the proof, see Appendix B. □

We employ Theorem 3 to prove that $\rho(\boldsymbol{x}_k, \mathscr{X}_{\text{bd}})$ is bounded from above by an exponential $\zeta \mathrm{e}^{-\alpha k} \rho(\boldsymbol{x}_0, \mathscr{X}_{\text{bd}})$ for some $\alpha > 0$ and $\zeta > 0$.

Theorem 10: *For the discrete load processor network system described above, the invariant set* $\mathscr{X}_{\text{bd}}$ *is exponentially stable in the large with respect to* $\boldsymbol{E}_B$.

Proof: The first condition of Theorem 3 is shown to hold in the proof of Theorem 9. We now show that the final condition of Theorem 3 holds.

We define a constant δ on which the proof will depend. For a given discrete load network, there is a constant $\delta_1 > 0$, such that if $e_{\alpha(i)}^{i,p(i)} \in e_k$ and $\alpha_j(i) > 0$ for some $j \in p(i)$, then $x_i' \geq x_{j^*} + \delta_1$, where $j^* \in p(i)$ and $x_{j^*} \leq x_j$ for all $j \in p(i)$. For the same discrete load network, there is also a constant $\delta_2 > 0$, such that if $(i, j) \in A$ and $x_i \neq x_j$, then $|x_i - x_j| \geq \delta_2$. Let $\delta = \min\{\delta_1, \delta_2, m\}$.

For $x_k \notin \mathscr{X}_{bd}$, there is $L^*(k) \subset L$ such that $L^*(k) = \{i\colon\ x_i \leq x_j, j \in L\}$. Because there must be at least one node in the network that is more heavily loaded than the rest of the nodes, we know that $|L^*(k)| \leq N - 1$.

Fix a time $k \geq 0$. There must be some $i \notin L^*(k)$ and some $j \in L^*(k)$ such that $(i, j) \in A$. According to the restrictions imposed by E_B, there is some time k_1, $k \leq k_1 < k + B$ such that $e_{\alpha(i)}^{i,p(i)} \in e_{k_1}$. Conditions (a)(ii) and (a)(iii) on $e_k \in g(x_k)$, along with the definition of δ, imply that either (a) $|L^*(k+1)| \leq |L^*(k)| - 1$ and $x'_q = x_j$ for all $q \in L^*(k+1)$ and all $j \in L^*(k)$; or (b) $x'_q \geq x_j + \delta$ for all $q \in L^*(k+1)$ and all $j \in L^*(k)$. In other words, either the number of least loaded nodes decreases by at least one or the smallest load increases by at least δ. Thus, because $|L^*(k)| \leq N - 1$, we can conclude that for $x_k \notin \mathscr{X}_{bd}$, $V(x_k) - V(x_{k+NB}) \geq \delta$. From (10), it is clear that $\sum_{i=1}^{N} x_i > \rho(x_k, \mathscr{X}_{bd})$. It is also clear that there is some $\zeta > 0$ such that

$$\zeta\delta > \sum_{i=1}^{N} x_i > \rho(x_k, \mathscr{X}_{bd})$$

Therefore, it follows that $V(x_k) - V(x_{k+NB}) > (1/\zeta)\rho(x_k, \mathscr{X}_{bd})$, which satisfies the final condition of Theorem 3. □

Remark 1: Notice that in the discrete load case, the rate of exponential convergence depends on N, B and ζ. As in the continuous load case, the smaller we make B, the faster we are guaranteed to converge. Unlike the continuous load case in which the guaranteed rate of convergence depends on tangible system constants R and S, in this case we have the peculiar dependence on ζ. It is less clear how to design for a small ζ than it is to design for a small R or S. If all the load blocks in the network have size M, then $\delta = M$ and $\zeta = \sum_{i=1}^{N} x_i/M$. However, if the load blocks are of various sizes then ζ must be calculated from a worst-case analysis. □

Remark 2: It can be shown if we change condition (a)(iii) to 'if $x_i - x_j > M$ for some $(i, j) \in A$ then $\alpha_j(i) > 0$ for some $(i, j) \in A$', thereby alleviating the nodes from scanning all of their neighbours to locate one of the least loaded, then $\mathscr{X}_{bd}$ is asymptotically stable in the large with respect to E_i and exponentially stable in the large with respect to E_B. Of course, the guaranteed rate of convergence will suffer under this less strict load passing condition.

4. The load balancing problem with delays

We now modify the model of the system to allow for delays in load transport and sensing as in Tsitsiklis and Bertsekas (1989). In this extended analysis, we no longer require that the real time between events e_k and e_{k+1} be greater than the greatest system transportation time plus the greatest system sensing time. In this sense, we allow a reduction of the degree of synchronicity forced upon the system. What we now require is that there exist $B > 0$ such that load passed at time k is received by time $k + B - 1$ and that for all $(i, j) \in A$ load which arrives at node j at time k' will be sensed by node i by time $k' + B - 1$.

Tsitsiklis and Bertsekas (1989) presented a proof for asymptotic stability and suggested a proof for geometric convergence. We take a different approach by studying the problem within the Lyapunov stability framework and proving asymptotic and exponential stability and providing a rate of convergence analysis. Our proof for asymptotic stability is different from the one of Tsitsiklis and Bertsekas (1989). Lemmas 1, 2 and 3 in our proof of exponential stability are adaptations of lemmas from the proof of Tsitsiklis and Bertsekas (1989). Lemma 4 in our proof of exponential stability provides sufficient conditions for exponential stability that are more general than those of Michel *et al.* (1992 a, b) and allow us to finish the proof for the delay case.

Let $\mathscr{X} = \Re^{(2N+|A|)\times B}$ be the set of states. (We use the term 'state' here for convenience. Strictly speaking, $x_k \in \mathscr{X}$ is not necessarily a 'state' in the conventional sense.) Every $\boldsymbol{x}_k \in \mathscr{X}$ is composed of three 'sub-states'. Let $\boldsymbol{x}_{n0} \in \Re^{N\times B}$ represent the loads of the N network nodes at times k, $k-1, \ldots, k-B+2$, $k-B+1$. The first column represents the loads of the nodes at time k, the second column represents the loads of the nodes at time $k-1$, and so on. Let $\boldsymbol{x}_{n1} \in \Re^{N\times B}$ represent the loads of the N network nodes at times $k-B$, $k-B-1, \ldots, k-2B+2$, $k-2B+1$. The first column represents the loads of the nodes at time $k-B$, the second column represents the loads of the nodes at time $k-B-1$, and so on. Let $\boldsymbol{x}_t \in \Re^{|A|\times B}$ represent all of the $|A|$ loads in transit between the N network nodes at times k, $k-1, \ldots, k-B+2$, $k-B+1$. The first column represents the loads in transit at time k, the second column represents the loads in transit at time $k-1$, and so on. Pictorially, the state $\boldsymbol{x}_k \in \mathscr{X}$ may be represented as

$$\boldsymbol{x}_k = \begin{bmatrix} \boldsymbol{x}_{n0} \\ \boldsymbol{x}_{n1} \\ \boldsymbol{x}_t \end{bmatrix} = \begin{bmatrix} \boldsymbol{x}_n \\ \boldsymbol{x}_t \end{bmatrix} \quad \text{where} \quad \boldsymbol{x}_n = \begin{bmatrix} \boldsymbol{x}_{n0} \\ \boldsymbol{x}_{n1} \end{bmatrix}$$

We also define

$$\boldsymbol{x}_s = \begin{bmatrix} \boldsymbol{x}_{n0} \\ \boldsymbol{x}_t \end{bmatrix}$$

so that the sum of the elements of any column of $\boldsymbol{x}_s$ is equal to the total network load. Let $\boldsymbol{x}_{n0}(k')$, $\boldsymbol{x}_{n1}(k')$, $\boldsymbol{x}_n(k')$, $\boldsymbol{x}_t(k')$, and $\boldsymbol{x}_s(k')$ be defined in the same manner as $\boldsymbol{x}_{n0}$, $\boldsymbol{x}_{n1}$, $\boldsymbol{x}_n$, $\boldsymbol{x}_t$, and $\boldsymbol{x}_s$, with the exception that the state from which they derive is $\boldsymbol{x}_{k'}$ instead of $\boldsymbol{x}_k$.

Let x_i denote the load of node $i \in L$ at time k, let x_i' denote the load of node $i \in L$ at time $k+1$, and let $x_i(k')$ denote the load of node $i \in L$ at time k'. Clearly, x_i is element i of the first column of $\boldsymbol{x}_k$, x_i' is element i of the first column of $\boldsymbol{x}_{k+1}$ and $x_i(k')$ is element i of the first column of $\boldsymbol{x}_{k'}$. Let $x_{i\to j}$ denote the load in transit from node i to node j at time k, and let $x_{i\to j}'$ denote the load in transit from node i to node j at time $k+1$, $(i, j) \in A$. Clearly, $x_{i\to j}$ is one of the last $|A|$ elements of the first column of $\boldsymbol{x}_k$, and $x_{i\to j}'$ is one of the last $|A|$ elements of the first column of $\boldsymbol{x}_{k+1}$. Let x_j^i be the perception by node i of the load of node j at time k, and let $x_j^i(k')$ be the perception by node i of the load of node j at time k'. Because of the restriction on the delay in sensing, x_j^i must be any one of the elements of row j of $\boldsymbol{x}_k$ (i.e. row j of $\boldsymbol{x}_{n0}$), and $x_j^i(k')$ must be any one of the elements of row j of $\boldsymbol{x}_{k'}$.

Let $e_{\alpha(i)}^{i\to p(i)}$ represent that node $i \in L$ passes load to its neighbours $m \in p(i)$ where $p(i) = \{j\colon \exists (i, j) \in A\}$. Let $\alpha(i) = (\alpha_j(i), \alpha_{j'}(i), \ldots, \alpha_{j''}(i))$ such that $j < j' < \cdots < j''$ and $j, j', \ldots, j'' \in p(i)$ and $\alpha_j \geqslant 0$ for all $j \in p(i)$; the size of the list is $|p(i)|$. For convenience, we will denote this list by $\alpha(i) = (\alpha_j(i)\colon j \in p(i))$. $\alpha_m(i)$ denotes the amount of load passed from $i \in L$ to $m \in p(i)$. Let

$\{e_{\alpha(i)}^{i\to p(i)}\}$ denote the set of all possible such load passes. Let $e_\beta^{j\leftarrow i}$ represent that node $j \in L$ receives $\beta \geq 0$ load from node i. Let $\{e_\beta^{j\leftarrow i}\}$ denote the set of all possible such load receptions. Let the set of events be described by

$$\mathscr{E} = \{\mathscr{P}(\{e_{\alpha(i)}^{i\to p(i)}\}) \cup \mathscr{P}(\{e_\beta^{j\leftarrow i})\} - \{\phi\}$$

As before, each event $e_k \in \mathscr{E}$ is defined as a set. Elements of e_k may represent either the passing of load by node $i \in L$ to its neighbouring nodes in the network or the reception of load by node $i \in L$. Once again, let the γ_{ij} for $(i, j) \in A$ be defined *a priori*.

Below, we specify g and f_e for $e_k \in g(\boldsymbol{x}_k)$.

(1) Event $e_k \in g(\boldsymbol{x}_k)$ if (a), (b), and (c) below hold.

(a) For all $e_{\alpha(i)}^{i\to p(i)} \in e_k$ where $\alpha(i) = (\alpha_j(i)\colon j \in p(i))$ it is the case that

(i) $\alpha_j(i) = 0$ if $x_i \leq x_j^i$ where $j \in p(i)$,

(ii) $0 \leq \sum_{m \in p(i)} \alpha_m(i) \leq x_i - (x_j^i + \alpha_j(i))$ for all $j \in p(i)$

such that $x_i \geq x_j^i$ and

(iii) $\alpha_{j^*}(i) \geq \gamma_{ij^*}(x_i - x_{j^*}^i)$ for some $j^* \in \{j\colon\ x_j^i \leq x_m^i$ for all $m \in p(i)\}$.

Condition (i) prevents load from being passed by node i to node j if node i is less heavily loaded than its perception of node j. Condition (ii) directly implies that $x_i - \sum_{m\in p(i)} \alpha_m(i) \geq x_j^i + \alpha_j(i)$. Thus, after the load $\alpha(i)$ has been passed, the remaining load of node i must be at least as large as $x_j^i + \alpha_j(i)$ for every node $j \in p(i)$ that was less heavily loaded than node i to begin with. Condition (iii) implies that if node i does not perceive itself as being load balanced with all of its neighbours, then i must pass a non-negligible portion of its load to some neighbour perceived to be least loaded, j^*.

(b) For all $e_\beta^{j\leftarrow i} \in e_k$ it is the case that $0 \leq \beta \leq x_{i\to j}$.

(c) If $e_{\alpha(i)}^{i\to p(i)} \in e_k$ where $\alpha(i) = (\alpha_j(i)\colon\ j \in p(i))$, then $e_{\delta(i)}^{i\to p(i)} \notin e_k$ where $\delta(i) = (\delta_j(i)\colon\ j \in p(i))$ if $\alpha_j(i) \neq \delta_j(i)$ for all $j \in p(i)$. Hence, in each valid event e_k, there must be a consistent definition of the load, $\alpha_j(i)$, to be passed from any node i to any other node j.

(d) If $e_\beta^{j\leftarrow i} \in e_k$, then $e_\delta^{j\leftarrow i} \notin e_k$ if $\beta \neq \delta$. Hence, in each valid event e_k, there must be a consistent definition of the load, β, received by any node j from any other node i, $(i, j) \in A$.

(2) If $e_k \in g(\boldsymbol{x}_k)$ then $f_{e_k}(\boldsymbol{x}_k) = \boldsymbol{x}_{k+1}$ where

$$x_i' = x_i - \sum_{\{j:j\in p(i), e_{\alpha(i)}^{i\to p(i)}\}} \alpha_j(i) + \sum_{\{m: e_\beta^{i\leftarrow m} \in e_k\}} \beta$$

$$x_{i\to j}' = x_{i\to j} + \sum_{\{j:j\in p(i), e_{\alpha(i)}^{i\to p(i)} \in e_k\}} \alpha_j(i) - \sum_{\{j: e_\beta^{j\leftarrow i} \in e_k\}} \beta$$

The load of node i at time $k+1$ is the load of node i at time k minus the total load passed by node i at time k plus the total load received by node i at time k. The load in transit from node i to any one of its neighbours, $j \in p(i)$, at time $k+1$ is the load in transit from node i to node j at time k plus the passed load, minus the received load.

Let $\boldsymbol{E}_v = \boldsymbol{E}$ be the set of valid event trajectories. We must further specify the set of allowed event trajectories, $\boldsymbol{E}_a \subset \boldsymbol{E}_v$. We define a partial event of type '$i\rightarrow$' to represent the passing of $\alpha(i)$ amount of load from $i \in L$ to its neighbours $p(i)$. A partial event of type $i\rightarrow$ will be denoted by $e^{i\rightarrow p(i)}$ and the occurrence of $e^{i\rightarrow p(i)}$ indicates that $i \in L$ attempts to balance its load with its neighbours further. We define a partial event of type '$j\leftarrow$' to represent the receiving of β amount of load by $j \in L$ from one of its neighbours in $p(j)$. Event e_k is composed of a set of partial events. For $\boldsymbol{E}_B \subset \boldsymbol{E}_v$, the following two conditions must hold for every $E \in \boldsymbol{E}_B$.

(1) There exists $B > 0$ such that in every substring $e_{k'}, e_{k'+1}, e_{k'+2}, \ldots, e_{k'+(B-1)}$ there is the occurrence of partial event $e^{i\rightarrow p(i)}$ for all $i \in L$ (i.e. for every $i \in L$ partial event $e^{i\rightarrow p(i)} \in e_k$ for some k, $k' \leq k \leq k' + B - 1$).

(2) For every i and k' such that $e_{\alpha(i)}^{i\rightarrow p(i)} \in e_{k'}$, there is $k' \leq k < k' + B$ such that $e_{\alpha_j(i)}^{j\leftarrow i} \in e_k$. This restriction mandates that load passed at time k' must be received intact by time $k' + B - 1$.

We want to define an invariant set such that any state $\boldsymbol{x}_k$ which is in the invariant set exhibits the following properties.

(i) The load in the nodes is perfectly balanced at time k.

(ii) There is no load in transit at time k.

(iii) At time k, every node has an accurate perception of the load of its neighbours.

Let $L' = \{1, 2, \ldots, 2N\}$, $G = \{1, 2, \ldots, B\}$ and $H = \{1, 2, \ldots, |A|\}$. If $\boldsymbol{y}$ is a matrix, let $(\boldsymbol{y})_{pq}$ denote the element in row p and column q of $\boldsymbol{y}$. Choose

$$\mathscr{X}_b = \{\boldsymbol{x}_k \in \mathscr{X}\colon (\boldsymbol{x}_n)_{ij} = (\boldsymbol{x}_n)_{pq} \text{ for all } i,\, p \in L' \text{ and } j,\, q \in G;\quad (\boldsymbol{x}_t)_{ij} = 0 \text{ for all } i \in H \text{ and } j \in G\} \tag{11}$$

Consider any $\boldsymbol{x}_k \in \mathscr{X}_b$. Because all elements of $\boldsymbol{x}_n$ are equal, the load in the nodes is perfectly balanced at time k. Because all elements of $\boldsymbol{x}_t$ are zero, there can be no load in transit at time k. Because the load at all nodes has been fixed since time $k - 2B + 1$, we are guaranteed that each node has an accurate perception of all of its neighbours at time k. Hence, $\mathscr{X}_b$ is an invariant set whose element states exhibit the required properties. Notice that the only $e_k \in g(\boldsymbol{x}_k)$, where $\boldsymbol{x}_k \in \mathscr{X}_b$, are ones such that all $e_{\alpha(i)}^{i\rightarrow p(i)} \in e_k$ have $\alpha(i) = (0, 0, \ldots, 0)$ and all $e_\beta^{j\leftarrow i} \in e_k$ have $\beta = 0$.

To study the ability of the system to redistribute load automatically to achieve balancing, we again employ a Lyapunov stability theoretic approach. Let $T = \{1, 2, \ldots, (2N + |A|)\}$. Choose

$$\rho(\boldsymbol{x}_k, \mathscr{X}_b) = \inf\{\max\{|(\boldsymbol{x}_k)_{ij} - (\bar{\boldsymbol{x}})_{ij}|\colon \text{for all } i \in T,\, j \in G\}\colon \bar{\boldsymbol{x}} \in \mathscr{X}_b\} \tag{12}$$

Theorem 11: *For the load processor network with delays as described above, the invariant set $\mathscr{X}_b$ is asymptotically stable in the large with respect to $\boldsymbol{E}_B$.*

Proof: For the proof, see Appendix C. □

We employ the exponential stability theorem to prove that $\rho(\boldsymbol{x}_k, \mathscr{X}_b)$ is bounded from above by an exponential $\zeta \mathrm{e}^{-\alpha k} \rho(\boldsymbol{x}_0, \mathscr{X}_b)$ for some $\alpha > 0$ and $\zeta > 0$.

Theorem 12: *For the load processor network with delays as described above, the invariant set $\mathscr{X}_b$ is exponentially stable in the large with respect to $\boldsymbol{E}_B$.*

Proof: For the proof, see Appendix D. □

In the proof in Appendix D, we show that

$$V(\boldsymbol{x}_{k-2B+1}) - V(\boldsymbol{x}_{k+3N^2B+2B}) \geq \frac{1}{B}\gamma^{2B-1}(\eta\gamma^{3N^2B+2B})^N \rho(\boldsymbol{x}_k, \mathscr{X}_b), \qquad (13)$$

for $k \geq 2B - 1$. Because (13) is valid only for $k \geq 2B - 1$, it should be apparent that $k = 2B - 1$ in our model is equivalent to $k = 0$ in Lemma 4 (from the proof in Appendix D). Hence, what we have shown via the proof is that for all $k \geq 2B - 1$

$$\rho(\boldsymbol{x}_k, \mathscr{X}_b) \leq \zeta \mathrm{e}^{-\alpha(k-2B+1)} \rho(\boldsymbol{x}_{2B-1}, \mathscr{X}_b)$$

for some $\alpha > 0$ and $\zeta > 0$. Of course, from the proof of asymptotic stability, we are assured that

$$\rho(\boldsymbol{x}_k, \mathscr{X}_b) \leq 2NB^2\left(2 + \frac{2B|A|}{N}\right)\rho(\boldsymbol{x}_0, \mathscr{X}_b)$$

for all $0 \leq k < 2B - 1$.

Remark 1: This remark will parallel Remark 2 that followed the proof of exponential stability for the non-delay, continuous load system. The value $(1/B\gamma^{2B-1})(\eta\gamma^{3N^2B+2B})^N$ from (13) is directly related to the α from the exponential overbounding function $\zeta \mathrm{e}^{-\alpha(k-2B+1)} \rho(\boldsymbol{x}_{2B-1}, \mathscr{X}_b)$. Thus, if speed of convergence is a design factor, then γ should be made as large as possible and N and B should be made as small as possible.

The condition $k' \geq k + 3NB$ in Lemma 2 (from the proof in Appendix D) is unnecessarily conservative. From the proof of Lemma 2, we see that the condition $k' \geq k + 3RB$, where $R = \max_i \{|p(i)|\} + 1$, is sufficient. Let S be the maximum number of arcs that must be spanned to reach any node $j \in L$ from any other node $i \in L$. Because every processor $i \in L$ is actually at a distance (in arcs) of S or less from every other processor $j \in L$, (13) can be validly written as

$$V(\boldsymbol{x}_{k-2B+1}) - V(\boldsymbol{x}_{k+3RSB+2B}) \geq \frac{1}{B}\gamma^{2B-1}(\eta\gamma^{3RSB+2B})^S \rho(\boldsymbol{x}_k, \mathscr{X}_b)$$

Therefore, convergence can once again be accelerated by designing for RS^2 as small as possible. □

Remark 2: The idea of virtual load works for the delay case similarly to the way in which it worked for the non-delay case. □

Remark 3: It is possible to extend the delay case to cover the possibility of discrete loads. The proofs would be similar in spirit to those found here. □

5. Concluding remarks

In § 1, we introduced a DES model and provided stability definitions and an approach to stability analysis for load balancing systems. In § 2 we studied the load balancing problem without delays in passing and sensing load. We proved in this case that a particular load redistribution policy is asymptotically and exponentially stable. We also generalized on the original non-delay load balancing system and proved that several of the generalized systems are asymptotically and exponentially stable. In § 3, we studied the load balancing problem and considered delays in passing and sensing load. We proved in this case that a particular load redistribution policy is asymptotically and exponentially stable.

While we have shown how to characterize and analyse stability properties of general load balancing systems with and without delays and have generalized the results of Tsitsiklis and Bertsekas (1989) in several ways, there still remains research to be done. For instance, in all of the load balancing problems considered in this paper (and in Tsitsiklis and Bertsekas 1989), it is assumed that no new load arrives at the network for processing and that no load is processed while the load is being balanced. Certainly the results indicate that if load arrives/departs while it is being balanced the system will continually seek to balance; but, in general, the systems will not possess the stability properties found in this paper. Clearly there is a need to characterize and analyse stability properties of the general load balancing problem with arrivals/departures. Another issue that needs to be addressed is how to generalize the dynamics of the load processor so that other types of load processing mechanisms can be considered. Finally, there is clearly a need to consider general load balancing problems in a stochastic framework.

Acknowledgments

This work was supported in part by an Ohio State University Seed Grant and by an Engineering Foundation Research Initiation Grant.

Appendix A—Proof of Theorem 5: Choose the same $V(x)$ as in (2). The first condition of Theorem 3 is shown to hold in the proof of asymptotic stability, Theorem 4. We now show that the second condition of Theorem 3 holds.

Let $\gamma = \min_{i,j}\{\gamma_{ij}\}$. For any $i \in L$ and $k \geq 0$, we know from condition (a)(ii) on $e^{i,p(i)}_{\alpha(i)} \in e_k$ and the definition of γ that if $e^{i,p(i)}_{\alpha(i)} \in e_k$ and $\alpha_j(i) > 0$ for some $j \in p(i)$, then $x'_i \geq x_j + \gamma(x_i - x_j)$ for some $j \in p(i)$. If $e^{i,p(i)}_{\alpha(i)} \notin e_k$ or $e^{i,p(i)}_{\alpha(i)} \in e_k$ and $\alpha(i) = (0, 0, \ldots, 0)$, then $x'_i = x_i$. It follows that in any case

$$x'_i \geq \min_i \{x_i\} + \gamma[x_i - \min_i \{x_i\}] \tag{A 1}$$

Thus, it is clear that $\min_i \{x_i\}$ is a non-decreasing function of k. We now show via induction on t that

$$x_i(k + t) \geq \min_i \{x_i\} + \gamma^t[x_i - \min_i \{x_i\}] \tag{A 2}$$

for all $t \geq 0$. Equation (14) is the statement of (A 2) for $t = 1$. Assume that (A 1) is true for an arbitrary t. If x_i denotes the load of $i \in L$ at time k, then according to (A 1):

$$
\begin{aligned}
x_i(k+t+1) &\geq \min_i \{x_i(k+t)\} + \gamma[x_i(k+t) - \min_i \{x_i(k+t)\}] \\
&\geq \min_i \{x_i\} + \gamma[x_i(k+t) - \min_i \{x_i\}] \\
&\geq \min_i \{x_i\} + \gamma[\min_i \{x_i\} + \gamma^t[x_i - \min_i \{x_i\}] - \min_i \{x_i\}] \\
&= \min_i \{x_i\} + \gamma^{t+1}[x_i - \min_i \{x_i\}]
\end{aligned}
$$

Thus, (A 2) must be valid for all $t \geq 0$.

Fix $i \in L$ and $k \geq 0$. We now show that the loads of all neighbours of i are bounded from below by a function of x_i for all k', $k' \geq k + NB$. Specifically, we will show that

$$x_j(k') \geq \min_i \{x_i\} + \gamma^{k'-k}[x_i - \min_i \{x_i\}] \text{ for all } k' \geq k + NB, \quad j \in p(i) \tag{A 3}$$

There are times $k_m \geq k$, $m \in \{1, 2, \ldots\}$, such that $e_{\alpha(i)}^{i,p(i)} \in e_{k_m}$, and for $k' \neq k_m$, $e_{\alpha(i)}^{i,p(i)} \notin e_{k'}$. According to the restriction on $E \in \boldsymbol{E}_B$, $k \leq k_1 < k + B$ and $k_{m-1} < k_m < k_{m+1} + B$ for all $m \in \{2, 3, \ldots\}$. Below, we investigate three cases that may occur at any time k_m. The different cases describe different possible relative load levels of node i and its neighbours. More than one case may apply to a given time k_m.

In the first case, there is time k_m, $m \in \{1, 2, \ldots\}$, and $j \in p(i)$ such that $x_j(k_m) < x_i(k_m)$ and $x_j(k_m) \leq x_{j'}(k_m)$ for all $j' \in p(i)$. According to condition (a)(iii) on $e_k \in g(\boldsymbol{x}_k)$, $\alpha_j(i) \geq \gamma[x_i(k_m) - x_j(k_m)]$. Utilizing this fact and applying (A 2) to x_i yields

$$
\begin{aligned}
x_j(k_m+1) &\geq x_j(k_m) + \gamma[x_i(k_m) - x_j(k_m)] \\
&\geq \min_i \{x_i\} + \gamma[x_i(k_m) - \min_i \{x_i\}] \\
&\geq \min_i \{x_i\} + \gamma[\min_i \{x_i\} + \gamma^{k_m-k}[x_i - \min_i \{x_i\}] - \min_i \{x_i\}] \\
&= \min_i \{x_i\} + \gamma^{k_m-k+1}[x_i - \min_i \{x_i\}]
\end{aligned}
$$

If we now apply (A 2) to x_j with $k = k_m + 1$ and $t = k' - k_m - 1$, it is clear that

$$
\begin{aligned}
x_j(k') &\geq \min_i \{x_i(k_m+1)\} + \gamma^{k'-k_m-1}[x_j(k_m+1) - \min_i \{x_i(k_m+1)\}] \\
&\geq \min_i \{x_i\} + \gamma^{k'-k_m-1}[\min_i \{x_i\} + \gamma^{k_m-k+1}[x_i - \min_i \{x_i\}] - \min_i \{x_i\}] \\
&\geq \min_i \{x_i\} + \gamma^{k'-k}[x_i - \min_i \{x_i\}] \text{ for all } k' \geq k_m + 1
\end{aligned} \tag{A 4}
$$

In the second case, there is time k_m, $m \in \{1, 2, \ldots\}$, and $j' \in p(i)$ such that at some time k_q, $1 \leq q < m$, $\alpha_{j'}(i) \geq \gamma[x_i(k_q) - x_{j'}(k_q)]$. In other words, at time k_q, node i passed at least $\gamma[x_i(k_q) - x_{j'}(k_q)]$ to node j'. We consider any $j \in p(i)$ such that $x_j(k_m) \geq x_{j'}(k_m)$. Applying (A 2) to x_j with $k = k_m$ and

$t = k' - k_m$ yields

$$x_j(k') \geq \min_i \{x_i(k_m)\} + \gamma^{k'-k_m}[x_j(k_m) - \min_i \{x_i(k_m)\}]$$

$$\geq \min_i \{x_i\} + \gamma^{k'-k_m}[x_{j'}(k_m) - \min_i \{x_i\}] \text{ for all } k' \geq k_m \qquad \text{(A 5)}$$

Clearly, (A 4) applies to node j' for all k', $k' \geq k_q + 1$. Because $k_m \geq k_q + 1$, we can substitute in (A 5) for $x_{j'}(k_m)$ from (A 4) to arrive at

$$x_j(k') \geq \min_i \{x_i\} + \gamma^{k'-k_m}[\min_i \{x_i\} + \gamma^{k_m-k}[x_i - \min_i \{x_i\}] - \min_i \{x_i\}]$$

$$\geq \min_i \{x_i\} + \gamma^{k'-k}[x_i - \min_i \{x_i\}] \text{ for all } k' \geq k_m \qquad \text{(A 6)}$$

In the third case, there is time k_m, $m \in \{1, 2, \ldots\}$, such that $x_i(k_m) \leq x_j(k_m)$ for all $j \in p(i)$ (i.e. all neighbours of node i are at least as heavily loaded as node i). In this case, for any $j \in p(i)$, it is clear from (A 2) with $k = k_m$ and $t = k' - k_m$ that

$$x_j(k') \geq \min_i \{x_i(k_m)\} + \gamma^{k'-k_m}[x_j(k_m) - \min_i \{x_i(k_m)\}]$$

$$\geq \min_i \{x_i\} + \gamma^{k'-k_m}[x_i(k_m) - \min_i \{x_i\}] \text{ for all } k' \geq k_m \qquad \text{(A 7)}$$

From (A 2) with $t = k_m - k$, it is also clear that

$$x_i(k_m) \geq \min_i \{x_i\} + \gamma^{k_m-k}[x_i - \min_i \{x_i\}] \qquad \text{(A 8)}$$

If follows then from (A 7) and (A 8) that

$$x_j(k') \geq \min_i \{x_i\} + \gamma^{k'-k_m}[\min_i \{x_i\} + \gamma^{k_m-k}[x_i - \min_i \{x_i\}] - \min_i \{x_i\}]$$

$$\geq \min_i \{x_i\} + \gamma^{k'-k}[x_i - \min_i \{x_i\}] \text{ for all } k' \geq k_m \qquad \text{(A 9)}$$

Now notice that at each time k_m, $m \in \{1, 2, \ldots\}$, it must be the case that exactly one of the following is true.

(i) There is at least one $j \in p(i)$ such that $\alpha_j(i) \geq \gamma[x_i(k_m) - x_j(k_m)]$ and at every time k_q, $q < m$, $\alpha_j(i) < \gamma[x_i(k_q) - x_j(k_q)]$ (i.e. node i passes a non-negligible amount of load at time k to at least one of its neighbours to which it has not passed a non-negligible amount of load since before time k_1).

(ii) For every $j \in p(i)$ such that $\alpha_j(i) \geq \gamma[x_i(k_m) - x_j(k_m)]$, there is some $q < m$ such that the load passed by processor i to processor j at time q satisfies $\alpha_j(i) \geq \gamma[x_i(k_q) - x_j(k_q)]$ (i.e. processor i passes a non-negligible amount of load only to neighbours $j \in p(i)$ to which it has not passed a non-negligible amount of load since time k_1).

(iii) For every $j \in p(i)$, $x_i(k_m) \leq x_j(k_m)$ (i.e. processor i cannot pass load to any of its neighbours $j \in p(i)$).

If (ii) is true, then the second case applies to all neighbours of i and (A 6) is valid for all $j \in p(i)$. Hence, because $k_N \leq k + NB$, if $m < N$, then (A 3) is valid. If (iii) is true, then the third case applies for all of the neighbours of i, and (A 9) is valid for all $j \in p(i)$. Hence, because $k_N < k + NB$, if $m < N$, then

(A 3) is valid. If (i) is true, the first case applies to all of the neighbours of i to which i passes a non-negligible amount of load, and (A 4) is valid for all $j \in p(i)$ for which $\alpha_j(i) \geq \gamma[x_i(k_m) - x_j(k_m)]$ is true. Because $|p(i)| < N$, either (ii) or (iii) must occur before k_N or (i) must occur for every k_m, $m \in \{1, 2, \ldots, N-1\}$. Therefore, (A 3) must be valid.

We now extend (A 3) to

$$x_j(k') \geq \min_i \{x_i\} + (\gamma^{k'-k})^l [x_i - \min_i \{x_i\}] \text{ for all } k' \geq k + lNB \quad \text{(A 10)}$$

where j is any node that is reachable from i by spanning l inter-processor connections (arcs $(i, j) \in A$). Equation (A 3) establishes the validity of (A 10) for $l = 1$. We assume (A 10) is valid for a general j at a distance l from i, and there must be some node $q \in p(j)$ such that q is at a distance $l + 1$ from i. Equation (A 3), applied to $q \in p(j)$, yields

$$\begin{aligned} x_q(k') &\geq \min_i \{x_i(k + lNB)\} + \gamma^{k'-(k+lNB)}[x_j(k + lNB) - \min_i \{x_i(k + lNB)\}] \\ &\geq \min_i \{x_i\} + \gamma^{k'-k}[x_j(k + lNB) - \min_i \{x_i\}] \\ &\quad \text{for all } k' \geq k + (l + 1)NB \end{aligned}$$

Substituting based on our inductive hypothesis

$$\begin{aligned} x_q(k') &\geq \min_i \{x_i\} + \gamma^{k'-k}[\min_i \{x_i\} + (\gamma^{k'-k})^l [x_i - \min_i \{x_i\}] - \min_i \{x_i\}] \\ &= \min_i \{x_i\} + (\gamma^{k'-k})^{l+1}[x_i - \min_i \{x_i\}] \text{ for all } k' \geq k + (l + 1)NB. \end{aligned}$$

Hence, (A 10) must be valid for all $l \geq 1$.

Because every processor in the network can be reached from i by spanning fewer than N arcs, (A 10) implies that

$$x_j(k') \geq \min_i \{x_i\} + (\gamma^{k'-k})^N [x_i - \min_i \{x_i\}] \quad \text{(A 11)}$$

for all $k' \geq k + N^2 B$, $j \in p(i)$. Because we have made no assumptions to the contrary, (A 11) is valid for any $i \in L$. Hence, we can replace x_i with $\max_i \{x_i\}$ and $j \in p(i)$ with $j \in L$ and (A 11) becomes

$$x_j(k') \geq \min_i \{x_i\} + (\gamma^{k'-k})^N [\max_i \{x_i\} - \min_i \{x_i\}]$$

for all $k' \geq k + N^2 B$, $j \in L$. It follows directly that

$$\min_i \{x_i(k')\} \geq \min_i \{x_i\} + (\gamma^{k'-k})^N [\max_i \{x_i\} - \min_i \{x_i\}] \quad \text{(A 12)}$$

for all $k' \geq k + N^2 B$.

Choose $k' = k + N^2 B$. For every $k \geq 0$, $\mathbf{x}_k \notin \mathscr{X}_b$, equations (5) and (A 12) imply that

$$\begin{aligned} V(\mathbf{x}_k) - V(\mathbf{x}_{k+N^2B}) &= \min_i \{x_i(k + N^2 B)\} - \min_i \{x_i\} \\ &\geq (\gamma^{N^2 B})^N [\max_i \{x_i\} - \min_i \{x_i\}] \\ &\geq \gamma^{N^3 B} \rho(\mathbf{x}_k, \mathscr{X}_b) \end{aligned} \quad \text{(A 13)}$$

The above equation satisfies the final condition of Theorem 3. □

Appendix B—Proof of Theorem 9: Choose

$$V(\boldsymbol{x}_k) = \begin{cases} \dfrac{1}{N}\sum_{i=1}^{N} x_i - \min_i \{x_i\}, & \boldsymbol{x}_k \notin \mathscr{X}_{\rm bd} \\ 0, & \boldsymbol{x}_k \in \mathscr{X}_{\rm bd} \end{cases} \tag{B 1}$$

Notice that for $\boldsymbol{x}_k \notin \mathscr{X}_{\rm bd}$, there must be two nodes i and j, $(i, j) \in A$, such that $x_i - x_j > M$. Because nodes i and j, $(i, j) \in A$, of any state $\bar{\boldsymbol{x}} \in \mathscr{X}_{\rm bd}$ must be such that $\bar{x}_i - \bar{x}_j \leqslant M$, it is clear from (10) that

$$\rho(\boldsymbol{x}_k, \mathscr{X}_{\rm bd}) \geqslant \tfrac{1}{2}\max\{x_i - x_j - M: (i, j) \in A\}$$

$$2\rho(\boldsymbol{x}_k, \mathscr{X}_{\rm bd}) \geqslant \max\{x_i - x_j - M: (i, j) \in A\} \triangleq \psi_1(\boldsymbol{x}_k) \tag{B 2}$$

According to (B 1), because $\max_i \{x_i\} \geqslant (1/N)\sum_{i=1}^{N} x_i$, $V(\boldsymbol{x}_k) \leqslant \max_i \{x_i\} - \min_i \{x_i\}$. Because there exists a network link between any two nodes that consist of fewer than N interprocessor connections, it must be true that $\max_i \{x_i\} - \min_i \{x_i\} \leqslant N \max\{x_i - x_j: (i, j) \in A\}$. Hence

$$V(\boldsymbol{x}_k) \leqslant N \max\{x_i - x_j: (i, j) \in A\}$$

$$\frac{1}{N} V(\boldsymbol{x}_k) \leqslant \max\{x_i - x_j: (i, j) \in A\} \triangleq \psi_2(\boldsymbol{x}_k) \tag{B 3}$$

Finally, notice that according to (10) and (B 1),

$$V(\boldsymbol{x}_k) = \rho(\boldsymbol{x}_k, \mathscr{X}_{\rm bd}) = 0, \ \boldsymbol{x}_k \in \mathscr{X}_{\rm bd} \tag{B 4}$$

We will find a constant $\eta \in (0, \infty)$ such that $\eta\rho(\boldsymbol{x}_k, \mathscr{X}_{\rm bd}) \geqslant V(\boldsymbol{x}_k)$ for all $\boldsymbol{x}_k \in \mathscr{X}$. From (B 4), we see that for all $\boldsymbol{x}_k \in \mathscr{X}_{\rm b}$, any value of η will suffice. Thus, we need only be concerned with $\boldsymbol{x}_k \notin \mathscr{X}_{\rm bd}$. Accordingly, we will find a constant $\phi \in (0, \infty)$ such that $\phi\psi_1(\boldsymbol{x}_k) \geqslant \psi_2(\boldsymbol{x}_k)$ for $\boldsymbol{x}_k \notin \mathscr{X}_{\rm bd}$. Notice from (B 2) and (B 3) that

$$\psi_1(\boldsymbol{x}_k) + M = \psi_2(\boldsymbol{x}_k)$$

and $\psi_1(\boldsymbol{x}_k) = \varepsilon$, $\varepsilon > 0$, implies that $\psi_2(\boldsymbol{x}_k) = \varepsilon + M$. From this, it is clear that $\phi \in [1, \infty)$. For very large values of ε, a value of ϕ close to unity will satisfy our requirement, but as ε approaches zero, the necessary value of ϕ approaches infinity. However, because the network contains a finite number of blocks, each of finite size, there must be some constant, ε_0, $M \geqslant \varepsilon_0 > 0$, such that for $\boldsymbol{x}_k \notin \mathscr{X}_{\rm bd}$, $\psi_1(\boldsymbol{x}_k) \geqslant \varepsilon_0$. Thus, if we choose $\phi = 2M/\varepsilon_0$, it is clear that if $\psi_1(\boldsymbol{x}_k) \geqslant M$, then

$$\phi\psi_1(\boldsymbol{x}_k) = \frac{2M}{\varepsilon_0}\psi_1(\boldsymbol{x}_k) \geqslant 2\psi_1(\boldsymbol{x}_k) \geqslant \psi_1(\boldsymbol{x}_k) + M = \psi_2(\boldsymbol{x}_k)$$

and if $\psi_1(\boldsymbol{x}_k) < M$, then

$$\phi\psi_1(\boldsymbol{x}_k) = \frac{2M}{\varepsilon_0}\psi_1(\boldsymbol{x}_k) \geqslant 2M \geqslant \psi_1(\boldsymbol{x}_k) + M = \psi_2(\boldsymbol{x}_k)$$

It follows that for all $\psi_1(\boldsymbol{x}_k) \geqslant \varepsilon_0$

$$\phi\psi_1(\boldsymbol{x}_k) \geqslant \psi_2(\boldsymbol{x}_k) \tag{B 5}$$

From (B 3)–(B 5), we see that

$$2\phi\rho(\boldsymbol{x}_k, \mathcal{X}_{bd}) \geq \phi\psi_1(\boldsymbol{x}_k) \geq \psi_2(\boldsymbol{x}_k) \geq \frac{1}{N}V(\boldsymbol{x}_k)$$

$$2N\phi\rho(\boldsymbol{x}_k, \mathcal{X}_{bd}) \geq V(\boldsymbol{x}_k) \text{ for all } \boldsymbol{x}_k \in \mathcal{X} \tag{B 6}$$

so that condition (ii) of the Theorem 1 is satisfied.

Notice from (10) that for $\boldsymbol{x}_k \notin \mathcal{X}_{bd}$,

$$\rho(\boldsymbol{x}_k, \mathcal{X}_{bd}) \leq \max_i \{x_i\} - \min_i \{x_i\} \tag{B 7}$$

Because

$$\frac{1}{N}\sum_{i=1}^{N} x_i \geq \frac{1}{N}[\max_i \{x_i\} + (N-1)\min_i \{x_i\}]$$

we see from (B 1) that

$$V(\boldsymbol{x}_k) \geq \frac{1}{N}[\max_i \{x_i\} + (N-1)\min_i \{x_i\}] - \min_i \{x_i\}$$

$$\geq \frac{1}{N}[\max_i \{x_i\} - \min_i \{x_i\}]$$

$$NV(\boldsymbol{x}_k) \geq \max_i \{x_i\} - \min_i \{x_i\} \tag{B 8}$$

Thus, from (B 4), (B 7), (B 8) we conclude that

$$NV(\boldsymbol{x}_k) \geq \rho(\boldsymbol{x}_k, \mathcal{X}_{bd}) \quad \text{for all} \quad \boldsymbol{x}_k \in \mathcal{X} \tag{B 9}$$

so that condition (i) of the Theorem 1 is satisfied.

Condition (iii) of the Theorem 1 is satisfied in exactly the same way as in the proof of Theorem 4 so that $\mathcal{X}_{bd}$ is stable in the sense of Lyapunov with respect to $\boldsymbol{E}_i$.

In order to show that $\mathcal{X}_{bd}$ is asymptotically stable in the large with respect to $\boldsymbol{E}_i$, we must show that for all $\boldsymbol{x}_0 \notin \mathcal{X}_{bd}$ and all E_k such that $E_k E \in \boldsymbol{E}_i(\boldsymbol{x}_0)$

$$V(X(\boldsymbol{x}_0, E_k, k)) \to 0 \quad \text{as} \quad k \to \infty \tag{B 10}$$

If $\boldsymbol{x}_k \notin \mathcal{X}_{bd}$, then there must be some lightest loaded node j^{**} (there may be more than one such node) and some other node i such that $(i, j^{**}) \in A$ and $x_i > x_{j^{**}}$. Because of the restrictions imposed by $\boldsymbol{E}_i$, we know that all the partial events are guaranteed to occur infinitely often. According to condition (a)(iii) on $e_k \in g(\boldsymbol{x}_k)$, each time partial event $e^{i,p(i)}$ occurs, $x_{j^{**}}$ is guaranteed to increase by m so that $x'_{j^{**}} \geq x_{j^{**}} + m$, and according to condition (a)(ii) on $e_k \in g(\boldsymbol{x}_k)$, x'_i is guaranteed to be greater than $x_{j^{**}}$. In fact, because the system is composed of a finite number of blocks, each of finite size, we know that there is some constant $\delta > 0$ such that $x'_i \geq x_{j^{**}} + \delta$. Thus, regardless of how many lightest loaded nodes there are, it is inevitable that eventually the overall lightest load in the network must increase. Hence, for every $k \geq 0$, there exists $k' > k$ such that $V(\boldsymbol{x}_{k'}) > V(\boldsymbol{x}_{k'+1})$ as long as $\boldsymbol{x}_{k'} \notin \mathcal{X}_{bd}$ so that (B 10) holds and $\mathcal{X}_{bd}$ is asymptotically stable in the large with respect to $\boldsymbol{E}_i$. □

Appendix C—Proof of Theorem 11: For convenience, we define some mathematical notation. If y is a matrix, then $\min\{y\}$ is equal to the minimum of all of the elements of y, $\max\{y\}$ is equal to the maximum of all of the elements of y, and $\sum y$ is equal to the sum of all of the elements of y. Further, let $(y)^i$ be column i of y.

Choose

$$V(x_k) = \frac{1}{NB}\sum x_s - \min\{x_n\} \tag{C 1}$$

Notice that $V(x_k)$ is the average load (total network load divided by N) minus the minimum load, taken over times $k-2B+1, \ldots, k-1, k$, at any node $i \in L$.

We first demonstrate that condition (ii) of Theorem 1 is satisfied by our choice of $\rho(x_k, \mathscr{X}_b)$ and $V(x_k)$.

It is clear from (1) and (12) that

$$\begin{aligned}\rho(x_k, \mathscr{X}_b) &\geqslant \max\{\tfrac{1}{2}(\max\{x_n\} - \min\{x_n\}), \max\{x_t\}\} \\ &\geqslant \max\{\tfrac{1}{2}(\max\{x_n\} - \min\{x_n\}), \tfrac{1}{2}\max\{x_t\}\} \\ &\geqslant \tfrac{1}{2}\max\{(\max\{x_n\} - \min\{x_n\}), \max\{x_t\}\}\end{aligned} \tag{C 2}$$

It is also clear that

$$\max\{x_k\} = \max\{\max\{x_n\}, \max\{x_t\}\}$$

We must consider two cases. If $\max\{x_n\} - \min\{x_n\} \geqslant \max\{x_t\}$, then

$$\max\{x_n\} \geqslant \max\{x_t\} \text{ and } \max\{x_n\} = \max\{x_k\}$$

It follows, then, from (C 2) that

$$2\rho(x_k, \mathscr{X}_b) \geqslant \max\{x_k\} - \min\{x_n\}$$

On the other hand, if $\max\{x_t\} \geqslant \max\{x_n\} - \min\{x_n\}$, then because we know that

$$\max\{x_t\} \geqslant \max\{x_t\} - \min\{x_n\}$$

and

$$\max\{x_k\} = \max\{\max\{x_t\}, \max\{x_n\}\}$$

it must be the case that

$$\max\{x_t\} \geqslant \max\{x_k\} - \min\{x_n\}$$

Once again, (C 2) implies that

$$2\rho(x_k, \mathscr{X}_b) \geqslant \max\{x_k\} - \min\{x_n\}$$

Thus, we can conclude that

$$2\rho(x_k, \mathscr{X}_b) \geqslant \max\{x_k\} - \min\{x_n\} \tag{C 3}$$

for all $x_k \notin \mathscr{X}_b$.

If $\max\{x_k\} \geqslant (1/NB)\sum x_s$, then (C 1) and (C 3) imply that

$$V(x_k) \leqslant \max\{x_k\} - \min\{x_n\} \leqslant 2\rho(x_k, \mathscr{X}_b) \tag{C 4}$$

However, it is possible that some load is in transit and that the load at the nodes is distributed such that $\max\{x_k\} < (1/NB)\sum x_s$. It is clearly true for all

k', $k - B < k' \leqslant k$, that the total load in transit is equal to the total system load minus the load at the nodes. Hence, if q is the column of $\boldsymbol{x}_k$ that contains the load at the nodes and in transit at time k', then

$$\begin{aligned}\sum(\boldsymbol{x}_t)^q &= \sum(\boldsymbol{x}_s)^q - \sum(\boldsymbol{x}_{n0})^q \\ &\geqslant \sum(\boldsymbol{x}_s)^q - N\max\{(\boldsymbol{x}_n)^q\} \\ &\geqslant \frac{1}{B}\sum\boldsymbol{x}_s - N\max\{\boldsymbol{x}_n\}\end{aligned}$$

However

$$|A|\max\{\boldsymbol{x}_t\} \geqslant \sum(\boldsymbol{x}_t)^q$$

for all k', $k - B < k' \leqslant k$. It follows that

$$\begin{aligned}\max\{\boldsymbol{x}_t\} &\geqslant \frac{1}{|A|}\left[\frac{1}{B}\sum\boldsymbol{x}_s - N\max\{\boldsymbol{x}_n\}\right] \\ &\geqslant \frac{1}{|A|}\left[\frac{1}{B}\sum\boldsymbol{x}_s - N\max\{\boldsymbol{x}_k\}\right] \qquad \text{(C 5)}\end{aligned}$$

Because of the maximum network transit time, this $\max\{\boldsymbol{x}_t\}$ resulted from at the most $B - 1$ load passes. Due to condition (a)(ii) on $e_k \in g(\boldsymbol{x}_k)$ and the maximum network sensing time, each of these load passes must have been smaller than $\max\{\boldsymbol{x}_n\} - \min\{\boldsymbol{x}_n\}$. Hence

$$\begin{aligned}\max\{\boldsymbol{x}_t\} &\leqslant (B - 1)(\max\{\boldsymbol{x}_n\} - \min\{\boldsymbol{x}_n\}) \\ &\leqslant B(\max\{\boldsymbol{x}_k\} - \min\{\boldsymbol{x}_n\}) \qquad \text{(C 6)}\end{aligned}$$

Equations (C 5) and (C 6) imply that

$$\max\{\boldsymbol{x}_k\} - \min\{\boldsymbol{x}_n\} \geqslant \frac{1}{B|A|}\left[\frac{1}{B}\sum\boldsymbol{x}_s - N\max\{\boldsymbol{x}_k\}\right] \qquad \text{(C 7)}$$

and (C 3) and (C 7) imply that

$$\begin{aligned}2\rho(\boldsymbol{x}_k, \mathscr{X}_b) &\geqslant \frac{1}{B|A|}\left[\frac{1}{B}\sum\boldsymbol{x}_s - N\max\{\boldsymbol{x}_k\}\right] \\ \frac{2B|A|}{N}\rho(\boldsymbol{x}_k, \mathscr{X}_b) &\geqslant \frac{1}{BN}\sum\boldsymbol{x}_s - \max\{\boldsymbol{x}_k\} \qquad \text{(C 8)}\end{aligned}$$

From (C 3) it is clear that $2\rho(\boldsymbol{x}_k, \mathscr{X}_b) + \min\{\boldsymbol{x}_n\} \geqslant \max\{\boldsymbol{x}_k\}$. Hence, from (C 8), it is clear that

$$\begin{aligned}\frac{2B|A|}{N}\rho(\boldsymbol{x}_k, \mathscr{X}_b) &\geqslant \frac{1}{BN}\sum\boldsymbol{x}_s - 2\rho(\boldsymbol{x}_k, \mathscr{X}_b) - \min\{\boldsymbol{x}_n\} \\ \left[\frac{2B|A|}{N} + 2\right]\rho(\boldsymbol{x}_k, \mathscr{X}_b) &\geqslant \frac{1}{BN}\sum\boldsymbol{x}_s - \min\{\boldsymbol{x}_n\} = V(\boldsymbol{x}_k) \qquad \text{(C 9)}\end{aligned}$$

Because (C 4) and (C 9) both bound $V(\boldsymbol{x}_k)$ from above, we can claim that $V(\boldsymbol{x}_k)$ is always bounded from above by the greater of the two bounds. Therefore, it is always true that

$$V(\boldsymbol{x}_k) \leqslant \left[2 + \frac{2B|A|}{N}\right]\rho(\boldsymbol{x}_k, \mathscr{X}_b) \tag{C 10}$$

so that condition (ii) of Theorem 1 is satisfied.

We now demonstrate that condition (i) of Theorem 1 is satisfied by our choice of $\rho(\boldsymbol{x}_k, \mathscr{X}_b)$ and $V(\boldsymbol{x}_k)$.

Notice that $\sum \boldsymbol{x}_s \geqslant \max\{\sum \boldsymbol{x}_{no}, \sum \boldsymbol{x}_{n1}\}$. It follows, then, from (C 1) that

$$V(\boldsymbol{x}_k) \geqslant \frac{1}{2NB}\sum \boldsymbol{x}_n - \min\{\boldsymbol{x}_n\} \tag{C 11}$$

In an analogous manner to the non-delay case, $\sum \boldsymbol{x}_n$ is minimized in terms of $\max\{\boldsymbol{x}_n\}$ and $\min\{\boldsymbol{x}_n\}$ when exactly one element of $\boldsymbol{x}_n$ is equal to $\max\{\boldsymbol{x}_n\}$ and the remaining elements of $\boldsymbol{x}_n$ are equal to $\min\{\boldsymbol{x}_n\}$. From this analysis and (47), we have that

$$\begin{aligned} V(\boldsymbol{x}_k) &\geqslant \frac{1}{2NB}[\max\{\boldsymbol{x}_n\} + (2NB - 1)\min\{\boldsymbol{x}_n\}] - \min\{\boldsymbol{x}_n\} \\ &= \frac{1}{2NB}[\max\{\boldsymbol{x}_n\} - \min\{\boldsymbol{x}_n\}] \end{aligned} \tag{C 12}$$

It is clear from (11) and (12) that

$$\rho(\boldsymbol{x}_k, \mathscr{X}_b) \leqslant \max\{(\max\{\boldsymbol{x}_n\} - \min\{\boldsymbol{x}_n\}), \max\{\boldsymbol{x}_t\}\} \tag{C 13}$$

We must consider two cases. First, consider $\max\{\boldsymbol{x}_n\} - \min\{\boldsymbol{x}_n\} \geqslant \max\{\boldsymbol{x}_t\}$. Then, according to (C 13),

$$\rho(\boldsymbol{x}_k, \mathscr{X}_b) \leqslant \max\{\boldsymbol{x}_n\} - \min\{\boldsymbol{x}_n\} \tag{C 14}$$

Equations (C 14) and (C 12) yield

$$V(\boldsymbol{x}_k) \geqslant \frac{1}{2NB}\rho(\boldsymbol{x}_k, \mathscr{X}_b) \tag{C 15}$$

Now, consider $\max\{\boldsymbol{x}_t\} > \max\{\boldsymbol{x}_n\} - \min\{\boldsymbol{x}_n\}$ so that

$$\rho(\boldsymbol{x}_k, \mathscr{X}_b) \leqslant \max\{\boldsymbol{x}_t\} \tag{C 16}$$

As before, the maximum load in transit at times $k - B + 1, \ldots, k - 1, k$, is the sum of at most $B - 1$ load passes, each of which must have been smaller than $\max\{\boldsymbol{x}_n\} - \min\{\boldsymbol{x}_n\}$. Hence,

$$\max\{\boldsymbol{x}_t\} \leqslant B[\max\{\boldsymbol{x}_n\} - \min\{\boldsymbol{x}_n\}] \tag{C 17}$$

A slight manipulation of (C 16) and (C 17), along with (C 12) yields

$$\begin{aligned} &\frac{1}{B}\rho(\boldsymbol{x}_k, \mathscr{X}_b) \leqslant \max\{\boldsymbol{x}_n\} - \min\{\boldsymbol{x}_n\} \\ \frac{1}{2NB^2}&\rho(\boldsymbol{x}_k, \mathscr{X}_b) \leqslant \frac{1}{2NB}[\max\{\boldsymbol{x}_n\} - \min\{\boldsymbol{x}_n\}] \leqslant V(\boldsymbol{x}_k) \end{aligned} \tag{C 18}$$

Because (C 15) and (C 18) both bound $V(x_k)$ from below, we can claim that $V(x_k)$ is always bounded from below by the lesser of the two bounds. Therefore, it is always true that

$$V(x_k) \geqslant \frac{1}{2NB^2}\rho(x_k, \mathscr{X}_b)$$

so that condition (i) of Theorem 1 is satisfied.

To satisfy the final condition of Theorem 1, we must show that $V(X(x_0, E_k, k))$ is a non-increasing function for $k \geqslant 0$ and all E_k such that $E_k E \in E_B(x_0)$. To see that this is the case, notice that once x_0 is specified, $V(x_k)$ varies only as $\min\{x_n\}$ varies. Clearly, what we must show is that $\min\{x_n\}$ is non-decreasing as a function of k. According to condition (a)(ii) on $e_k \in g(x_k)$, if $e_{\alpha(i)}^{i \to p(i)} \in e_k$ and $q(i) = \{j: j \in p(i)$ and $x_i \geqslant x_j^i\}$ then $x_i' \geqslant x_j^i$ for all $j \in q(i)$. In words, no node can pass so much load that its load level drops below its pre-pass perception of the load level of any node that it passed to. Therefore, because we are assured that $x_j^i \geqslant \min\{x_n\}$ for all $i \in L$ and $j \in p(i)$, $x_i' \geqslant \min\{x_n\}$ for all $i \in L$. Hence, $\min\{x_n\}$ is a non-decreasing function of k. Thus, condition (iii) of Theorem 1 is satisfied and $\mathscr{X}_b$ is stable in the sense of Lyapunov with respect to E_B.

In order to show that $\mathscr{X}_b$ is asymptotically stable in the large with respect to E_B, we must show that for all $x_0 \notin \mathscr{X}_b$ and all E_k such that $E_k E \in E_B(x_0)$,

$$V(X(x_0, E_k, k)) \to 0 \quad \text{as } k \to \infty \tag{C 19}$$

If $x_k \notin \mathscr{X}_b$, then x_{k+1} will represent a change of the load levels of all of the nodes included in some non-empty subset of L. Any change in the load of node $i \in L$ that is not positive must be due to the passing of load by node i at time k. In fact, from conditions (a)(ii) and (a)(iii) on $e_k \in g(x_k)$, we have that if $e_{\alpha(i)}^{i \to p(i)} \in e_k$ then

$$\begin{aligned} x_i' &\geqslant x_j^i + \gamma_{ij}[x_i - x_j^i] \text{ for some } j \text{ such that } \alpha_j(i) > 0 \\ &\geqslant \min\{x_n\} + \gamma[x_i - \min\{x_n\}] \end{aligned} \tag{C 20}$$

where $\gamma = \min_{(i,j) \in A}\{\gamma_{ij}\}$. Notice that (C 20) is valid even if $e_{\alpha(i)}^{i \to p(i)} \notin e_k$ or $\alpha(i) = (0, 0, \ldots, 0)$. Thus, for any time $k' > k$, $x_i(k') \geqslant \min\{x_n\}$. Again notice that $\min\{x_n\}$ is a non-decreasing function of k.

The question now becomes whether or not the passing of load is guaranteed to increase $\min\{x_n\}$. Employing (C 20) and the fact that $\min\{x_n\}$ is a non-decreasing function of k, we will use induction to show that

$$x_i(k + l) \geqslant \min\{x_n\} + (\gamma)^l[x_i - \min\{x_n\}] \text{ for all } i \in L \tag{C 21}$$

The case of $l = 1$ is simply (C 20). Assume that (C 21) is valid for some general l. Then, from (C 20) and (C 21),

$$\begin{aligned} x_i(k + l + 1) &\geqslant \min\{x_n(k + l)\} + \gamma[x_i(k + l) - \min\{x_n(k + l)\}] \\ &\geqslant \min\{x_n\} + \gamma[x_i(k + l) - \min\{x_n\}] \\ &\geqslant \min\{x_n\} + \gamma[\min\{x_n\} + (\gamma)^l[x_i - \min\{x_n\}] - \min\{x_n\}] \\ &= \min\{x_n\} + (\gamma)^{l+1}[x_i - \min\{x_n\}] \text{ for all } i \in L \end{aligned}$$

Thus, we have shown that (C 21) is valid in general.

Fix a time k such that $\boldsymbol{x}_k \notin \mathscr{X}_b$. If $x_i > \min\{\boldsymbol{x}_n\}$ for all $i \in L$, then

$$x_i(k+m) \geqslant \min\{\boldsymbol{x}_n\} + (\gamma)^{2B-1}[x_i - \min\{\boldsymbol{x}_n\}] \tag{C 22}$$

for all $i \in L$ and $m \in \{1, 2, \ldots, 2B-1\}$. From (C 22) and the definition of the state, it is clear that

$$\begin{aligned}\min\{\boldsymbol{x}_n(k+2B-1)\} &\geqslant \min\{\boldsymbol{x}_n\} + (\gamma)^{2B-1}[\min_i\{x_i\} - \min\{\boldsymbol{x}_n\}] \\ &> \min\{\boldsymbol{x}_n\}\end{aligned} \tag{C 23}$$

Let $L_* \subset L$ be the set of all i such that $x_i = \min\{\boldsymbol{x}_n\}$. It is possible that $|L_*| > 0$. Because $x_i(k') \geqslant \min\{\boldsymbol{x}_k\}$ for all $k' > k$, if $x_i = \min\{\boldsymbol{x}_n\}$, then $x_i(k-m) = \min\{\boldsymbol{x}_n\}$ for all $m \in \{1, 2, \ldots, 2B-1\}$. Thus, for any two nodes i and j such that $(i, j) \in A$ and $j \in L_*$, $x_j^i = \min\{\boldsymbol{x}_k\}$. According to the restrictions imposed on valid event strings by $\boldsymbol{E}_B$, there must be times k' and k'', $k < k' \leqslant k''$, $k'' < k + 2B$, such that $e_{\alpha_j(i)}^{i \to p(i)} \in e_{k'}$ and $e_{\alpha_j(i)}^{j \leftarrow i} \in e_{k''}$ for some $i \in L$ and $j \in L_*$ such that $(i, j) \in A$. Because $|L_*| < N$, the above passing and receiving scenario may have to transpire $N-1$ times to ensure that $x_j(k_1) > \min\{\boldsymbol{x}_n\}$ for all $j \in L_*$ and for some $k_1 \geqslant k$. It is apparent that for $k_1 = k + 2NB$, $x_j(k_1) > \min\{\boldsymbol{x}_n\}$ for all $j \in L_*$. Let $L^* \subset L$ such that $L^* \cup L_* = L$ and $L^* \cup L_* = \phi$. From (C 21),

$$\min_{i \in L^*}\{x_i(k_1)\} \geqslant \min\{\boldsymbol{x}_n\} + (\gamma)^{2NB}[\min_{i \in L^*}\{x_i\} - \min\{\boldsymbol{x}_n\}]$$

For any $j \in L_*$ that receives load at time $k'' < k_1$ that was passed at time $k' \geqslant k$, we have from the above equation, the fact that $x_j^i(k') = \min\{\boldsymbol{x}_n\}$, and (C 21) that

$$\begin{aligned}x_j(k'') &\geqslant x_j(k''-1) + \gamma[\min_{i \in L^*}\{x_i(k')\} - \min\{\boldsymbol{x}_n\}] \\ &\geqslant \min\{\boldsymbol{x}_n\} + \gamma[\min_{i \in L^*}\{x_i(k')\} - \min\{\boldsymbol{x}_n\}] \\ &\geqslant \min\{\boldsymbol{x}_n\} + \gamma[\min\{\boldsymbol{x}_n\} + (\gamma)^{k'-k}[\min_{i \in L^*}\{x_i\} - \min\{\boldsymbol{x}_n\}] - \min\{\boldsymbol{x}_n\}] \\ &\geqslant \min\{\boldsymbol{x}_n\} + (\gamma)^{k'-k+1}[\min_{i \in L^*}\{x_i\} - \min\{\boldsymbol{x}_n\}]\end{aligned}$$

From (C 21),

$$\begin{aligned}x_j(k_1) &\geqslant \min\{\boldsymbol{x}_n\} + (\gamma)^{k_1-k''}[x_j(k'') - \min\{\boldsymbol{x}_n\}] \\ &\geqslant \min\{\boldsymbol{x}_n\} + (\gamma)^{k_1-k''}[\min\{\boldsymbol{x}_n\} + \\ &\qquad (\gamma)^{k'-k+1}[\min_{i \in L^*}\{x_i\} - \min\{\boldsymbol{x}_n\}] - \min\{\boldsymbol{x}_n\}] \\ &\geqslant \min\{\boldsymbol{x}_n\} + (\gamma)^{2NB}[\min_{i \in L^*}\{x_i\} - \min\{\boldsymbol{x}_n\}]\end{aligned}$$

Therefore

$$\min_i\{x_i(k+2NB)\} \geqslant \min\{\boldsymbol{x}_n\} + (\gamma)^{2NB}[\min_{i \in L^*}\{x_i\} - \min\{\boldsymbol{x}_n\}]$$

and

$$\begin{aligned}\min\{\boldsymbol{x}_n(k+2(N+1)B)\} &\geqslant \min\{\boldsymbol{x}_n\} + (\gamma)^{2NB}[\min_{i \in L^*}\{x_i\} - \min\{\boldsymbol{x}_n\}] \\ &\geqslant \min\{\boldsymbol{x}_n\}\end{aligned} \tag{C 24}$$

Equations (C 23) and (C 24) and the definition of $V(\boldsymbol{x}_k)$ imply that

$$V(\boldsymbol{x}_k) - V(\boldsymbol{x}_{k+2(N+1)B}) \geq (\gamma)^{2NB}[\min_{i \in L^*}\{\boldsymbol{x}_i\} - \min\{\boldsymbol{x}_n\}] > 0 \qquad \text{(C 25)}$$

Therefore, (C 19) holds and $\mathscr{X}_b$ is asymptotically stable in the large with respect to $\boldsymbol{E}_B$. □

Appendix D—Proof of Theorem 12: Lemmas 1, 2 and 3 are adaptations of lemmas from the proof of Tsitsiklis and Bertsekas (1989).

Fix processor i and time k. For any $j \in p(i)$ and any time $k' > k$, we will say that system condition $E_j(k')$ occurs if

$$\text{(i)} \qquad x_j^i(k') < \min\{\boldsymbol{x}_n\} + \frac{\gamma}{2}\gamma^{k'-k}[x_i - \min\{\boldsymbol{x}_n\}] \qquad \text{(D 1)}$$

$$\text{(ii)} \qquad e_{\alpha(i)}^{i \to p(i)} \in e_{k'}, \ \alpha_j(i) \geq \gamma[x_i(k') - x_j^i(k')] \qquad \text{(D 2)}$$

Lemma 1: *If $j \in p(i)$, $k_1 > k$, $e_{\alpha(i)}^{i \to p(i)} \in e_{k_1}$, and $E_j(k_1)$ occurs, then $E_j(\bar{k})$ does not occur for $\bar{k} \geq k_1 + 2B$.*

Proof: Suppose $k_1 \geq k$, $e_{\alpha(i)}^{i \to p(i)} \in e_{k_1}$, and $E_j(k_1)$ occurs. From (C 21)

$$x_i(k_1) \geq \min\{\boldsymbol{x}_n\} + \gamma^{k_1-k}[x_i - \min\{\boldsymbol{x}_n\}] \qquad \text{(D 3)}$$

Subtracting (D 1) with $k' = k_1$ from (D 3) yields

$$x_i(k_1) - x_j^i(k_1) \geq \left(1 - \frac{\gamma}{2}\right)\gamma^{k_1-k}[x_i - \min\{\boldsymbol{x}_n\}]$$
$$\geq \tfrac{1}{2}\gamma^{k_1-k}[x_i - \min\{\boldsymbol{x}_n\}]$$

If we let $k' = k_1$, (D 2) yields

$$\alpha_j(i) \geq \gamma[x_i(k_1) - x_j^i(k_1)]$$
$$\geq \frac{\gamma}{2}\gamma^{k_1-k}[x_i - \min\{\boldsymbol{x}_n\}] \qquad \text{(D 4)}$$

According to the restrictions placed on valid event strings by $\boldsymbol{E}_a$, processor j will receive load $\alpha_j(i)$ at some time k_2, $k_1 \leq k_2 < k_1 + B$. Hence

$$x_j(k_2 + 1) \geq x_j(k_2) - \zeta + \alpha_j(i)$$
$$\geq \min\{\boldsymbol{x}_n(k_2)\} + \alpha_j(i)$$

where ζ is the total load (which may be zero) passed by processor j at time k_2. Using (D 4) this becomes

$$x_j(k_2 + 1) \geq \min\{\boldsymbol{x}_n\} + \alpha_j(i)$$
$$\geq \min\{\boldsymbol{x}_n\} + \frac{\gamma}{2}\gamma^{k_1-k}[x_i - \min\{\boldsymbol{x}_n\}]$$
$$\geq \min\{\boldsymbol{x}_n\} + \frac{\gamma}{2}\gamma^{k_2+1-k}[x_i - \min\{\boldsymbol{x}_n\}]$$

Using (C 21) it follows that for all $k_3 > k_2 + 1$,

$$\begin{aligned} x_j(k_3) &\geqslant \min\{x_n\} + \gamma^{k_3-k_2-1}[x_j(k_2+1) - \min\{x_n\}] \\ &\geqslant \min\{x_n\} + \gamma^{k_3-k_2-1}[\min\{x_n\} + \\ &\qquad \frac{\gamma}{2}\gamma^{k_2+1-k}[x_i - \min\{x_n\}] - \min\{x_n\}] \qquad \text{(D 5)} \\ &\geqslant \min\{x_n\} + \gamma^{k_3-k}[x_i - \min\{x_n\}] \qquad \text{(D 6)} \end{aligned}$$

Because $k_2 < k_1 + B$, (D 6) is valid for all $k_3 \geqslant k_1 + B$.

Let $k_4 \geqslant k_1 + 2B$ such that $e_{\alpha(i)}^{i \to p(i)} \in e_{k_4}$. According to the maximum system sensing time, there is some time k_5, $k_4 \geqslant k_5 > k_4 - B$, such that $x_j^i(k_4) = x_j(k_5)$. Equation (D 6) is valid at time k_5 and yields

$$\begin{aligned} x_j^i(k_4) - \min\{x_n\} = x_j(k_5) - \min\{x_n\} &\geqslant \frac{\gamma}{2}\gamma^{k_5-k}[x_i - \min\{x_n\}] \\ &\geqslant \frac{\gamma}{2}\gamma^{k_4-k}[x_i - \min\{x_n\}]. \end{aligned}$$

Therefore, equation (D 1) does not hold at k_4 and $E_j(k_4)$ does not occur. □

Lemma 2: *There exists some $\eta > 0$ such that for any $i \in L$, $k \geqslant 0$, $j \in p(i)$ and any $k' \geqslant k + 3NB$, we have*

$$x_j(k') \geqslant \min\{x_n\} + \eta\gamma^{k'-k}[x_i - \min\{x_n\}]$$

Proof: Fix i and k. Let $k_1, \ldots, k_N$ be times such that $e_{\alpha(i)}^{i \to p(i)} \in e_{k_m}$ and $k_{m-1} + 2B < k_m \leqslant k_{m-1} + 3B$ for all $m \in \{1, 2, \ldots, N\}$. According to Lemma 1, if $j \in p(i)$ and $m \neq l$, then $E_j(k_m)$ and $E_j(k_l)$ cannot both occur. Thus, there is some k_m, $m \in \{1, 2, \ldots, N\}$, such that $E_j(k_m)$ does not occur for any $j \in p(i)$. According to condition (a)(iii) on $e_k \in g(x_k)$, (D 2) must be valid for some j^* such that $x_{j^*}^i(k_m) \leqslant x_j^i(k_m)$ for all $j \in p(i)$. Because $E_j(k_m)$ does not occur, (D 3) is violated for $j = j^*$. It follows that for all $j \in p(i)$

$$x_j^i(k_m) \geqslant \min\{x_n\} + \frac{\gamma}{2}\gamma^{k_m-k}[x_i - \min\{x_n\}] \qquad \text{(D 7)}$$

According to the maximum system sensing time, there is some time k_1, $k_m \geqslant k_1 > k_m - B$, such that $x_j^i(k_m) = x_j(k_1)$. For any k_2, $k_2 \geqslant k + 3NB$, we have $k_2 \geqslant k_m \geqslant k_1$, and (C 21) yields

$$x_j(k_2) \geqslant \min\{x_n\} + \gamma^{k_2-k_1}[x_j(k_1) - \min\{x_n\}]$$

Realizing that $x_j(k_1) = x_j^i(k_m)$, we employ (D 7) to conclude that

$$\begin{aligned} x_j(k_2) &\geqslant \min\{x_n\} + \gamma^{k_2-k_1}[\min\{x_n\} + \frac{\gamma}{2}\gamma^{k_m-k}[x_i - \min\{x_n\}] - \min\{x_n\}] \\ &\geqslant \min\{x_n\} + \frac{\gamma}{2}\gamma^{(k_2-k)+(k_m-k_1)}[x_i - \min\{x_n\}]] \\ &\geqslant \min\{x_n\} + \frac{\gamma}{2}\gamma^{k_2-k}\gamma^{B}[x_i - \min\{x_n\}]] \end{aligned}$$

This proves Lemma 1 with $\eta = \gamma^{B+1}/2$. □

Lemma 3: *For any $i \in L$, any $k \geqslant 0$, any $j \in L$ that can be reached from i by traversing l arcs, and for any $k' \geqslant k + 3lNB$, we have*

$$x_j(k') \geqslant \min\{\boldsymbol{x}_n\} + (\eta\gamma^{k'-k})^l[x_i - \min\{\boldsymbol{x}_n\}]$$

Proof: Lemma 2 establishes Lemma 3 for $l = 1$. Assume that Lemma 3 is true for every j at a distance of l from i. Assume m is at distance $l + 1$ from i. Then $m \in p(j)$ for some j at a distance l from i. It follows from our inductive hypothesis that

$$x_j(k + 3lNB) \geqslant \min\{\boldsymbol{x}_n\} + (\eta\gamma^{3lNB})^l[x_i - \min\{\boldsymbol{x}_n\}]$$

If we apply Lemma 2 to processor $m \in p(j)$ at time $k_1 \geqslant k + 3lNB + 3NB$, we see that

$$\begin{aligned}
x_m(k_1) &\geqslant \min\{\boldsymbol{x}_n\} + \eta\gamma^{k_1-k-3lNB}[x_j(k + 3lNB) - \min\{\boldsymbol{x}_n\}] \\
&\geqslant \min\{\boldsymbol{x}_n\} + \eta\gamma^{k_1-k-3lNB}[\min\{\boldsymbol{x}_n\} + \\
&\quad (\eta\gamma^{3lNB})^l[x_i - \min\{\boldsymbol{x}_n\}] - \min\{\boldsymbol{x}_n\}] \\
&\geqslant \min\{\boldsymbol{x}_n\} + \eta\gamma^{k_1-k-3lNB}(\eta\gamma^{3lNB})^l[x_i - \min\{\boldsymbol{x}_n\}] \\
&\geqslant \min\{\boldsymbol{x}_n\} + \eta\gamma^{k_1-k}(\eta\gamma^{k_1-k})^l[x_i - \min\{\boldsymbol{x}_n\}] \\
&\geqslant \min\{\boldsymbol{x}_n\} + (\eta\gamma^{k_1-k})^{l+1}[x_i - \min\{\boldsymbol{x}_n\}]
\end{aligned}$$

Hence, the induction is complete and we have proven Lemma 3. □

Fix $i \in L$ and $k \geqslant 2B - 1$. Because every processor is at a distance of less than N from i, Lemma 3 yields

$$x_j(k') \geqslant \min\{\boldsymbol{x}_n\} + (\eta\gamma^{3N^2B+2B})^N[x_i - \min\{\boldsymbol{x}_n\}]$$

for all j, for all $k' \in [k + 3N^2B, k + 3N^2B + 2B]$. Hence

$$\min\{\boldsymbol{x}_n(k + 3N^2B + 2B)\} \geqslant \min\{\boldsymbol{x}_n\} + (\eta\gamma^{3N^2B+2B})^N[x_i - \min\{\boldsymbol{x}_n\}]$$

This relation is true for all $i \in L$. Thus

$$\begin{aligned}
\min\{\boldsymbol{x}_n(k + 3N^2B + 2B)\} &\geqslant \min\{\boldsymbol{x}_n\} + (\eta\gamma^{3N^2B+2B})^N[\max_i\{x_i\} - \min\{\boldsymbol{x}_n\}] \\
&\geqslant \min\{\boldsymbol{x}_n(k - 2B + 1)\} + (\eta\gamma^{3N^2B+2B})^N \\
&\quad \times [\max_i\{x_i\} - \min\{\boldsymbol{x}_n(k - 2B + 1)\}] \qquad \text{(D 8)}
\end{aligned}$$

Invoking the definition of the state, it is clear that

$$\max\{\boldsymbol{x}_n\} = \max_i\{x_i(k - m)\} \text{ for some } m \in \{0, 1, \ldots, 2B - 1\}.$$

Equation (C 21) and the above equation imply that

$$\begin{aligned}
\max_i\{x_i\} &\geqslant \min\{\boldsymbol{x}_n(k - m)\} + \gamma^m[\max_i\{x_i(k - m)\} - \min\{\boldsymbol{x}_n(k - m)\}] \\
&\geqslant \min\{\boldsymbol{x}_n(k - m)\} + \gamma^m[\max\{\boldsymbol{x}_n\} - \min\{\boldsymbol{x}_n(k - m)\}]
\end{aligned}$$

Manipulating further, we obtain

$$\max_i \{x_i\} - \min \{x_n(k-m)\} \geqslant \gamma^m[\max \{x_n\} - \min \{x_n(k-m)\}]$$

$$\begin{aligned}\max_i \{x_i\} - \min \{x_n(k-2B+1)\} &\geqslant \gamma^{2B-1}[\max \{x_n\} - \min \{x_n(k-m)\}] \\ &\geqslant \gamma^{2B-1}[\max \{x_n\} - \min \{x_n\}] \qquad \text{(D 9)}\end{aligned}$$

From (D 8) and (D 9), it follows that

$$\begin{aligned}\min \{x_n(k+3N^2B+2B)\} \geqslant& \min \{x_n(k-2B+1)\} \\ &+ \gamma^{2B-1}(\eta\gamma^{3N^2B+2B})^N[\max \{x_n\} - \min \{x_n\}]\end{aligned}$$

and

$$\begin{aligned}\min \{x_n(k+3N^2B+2B)\} - \min \{x_n(k-2B+1)\} \geqslant& \\ \gamma^{2B-1}(\eta\gamma^{3N^2B+2B})^N[\max \{x_n\} - \min \{x_n\}]\end{aligned}$$

From (C 1) it is apparent that

$$V(x_{k+3N^2B+2B}) = \frac{1}{NB}\sum x_s - \min \{x_n(k+3N^2B+2B)\}$$

and

$$V(x_{k-2B+1}) = \frac{1}{NB}\sum x_s - \min \{x_n(k-2B+1)\}$$

Clearly, then

$$\begin{aligned}V(x_{k-2B+1}) - V(x_{k+3N^2B+2B}) &= \min \{x_n(k+3N^2B+2B)\} \\ &\quad - \min \{x_n(k-2B+1)\} \\ &\geqslant \gamma^{2B-1}(\eta\gamma^{3N^2B+2B})^N \cdot \\ &\quad [\max \{x_n\} - \min \{x_n\}] \qquad \text{(D 10)}\end{aligned}$$

for all $k \geqslant 2B-1$. In the proof of asymptotic stability it is shown that

$$\max \{x_n\} - \min \{x_n\} \geqslant \frac{1}{B}\rho(x_k, \mathscr{X}_b)$$

for all $x_k \notin \mathscr{X}_b$. Hence, (D 10) becomes

$$V(x_{k-2B+1}) - V(x_{k+3N^2B+2B}) \geqslant \frac{1}{B}\gamma^{2B-1}(\eta\gamma^{3N^2B+2B})^N\rho(x_k, \mathscr{X}_b) \quad \text{(D 11)}$$

Lemma 4: *The closed, invariant set $\mathscr{X}_m \in \mathscr{X}$ is exponentially stable with respect to E_a if there exists a V defined on $S(\mathscr{X}_m; r)$, constants c_1, c_2, $c_3 > 0$, and constants D, D_1 such that $D > D_1$ and D, $D_1 \in N$ such that*

(i) $V(x_{k+1}) \leqslant V(x_k)$ *for all* $k \geqslant 0$

(ii) $c_1\rho(x, \mathscr{X}_m) \leqslant V(x) \leqslant c_2\rho(x, \mathscr{X}_m)$ *for all* $x \in S(\mathscr{X}_m; r)$

(iii) $V(x_k) - V(x_{k+D}) \geqslant c_3\rho(x_{k+D_1}, \mathscr{X}_m)$ *for all* $k \geqslant 0$

Proof: Conditions (i), (ii) and (iii) imply that

$$\begin{aligned} V(\boldsymbol{x}_k) - V(\boldsymbol{x}_{k+D}) &\geq c_3 \rho(\boldsymbol{x}_{k+D_1}, \mathscr{X}_m) \\ &\geq \frac{c_3}{c_2} V(\boldsymbol{x}_{k+D_1}) \\ &\geq \frac{c_3}{c_2} V(\boldsymbol{x}_{k+D}) \end{aligned} \tag{D 12}$$

We now show via induction that for all integers $m \geq 0$

$$V(\boldsymbol{x}_{mD}) \leq \left(1 + \frac{c_3}{c_2}\right)^{-m} V(\boldsymbol{x}_0) \tag{D 13}$$

Our induction hypothesis is that (D 13) is valid for some $m \geq 0$. From (D 12) and our induction hypothesis it follows that

$$V(\boldsymbol{x}_{mD}) - V(\boldsymbol{x}_{(m+1)D}) \geq \frac{c_3}{c_2} V(\boldsymbol{x}_{(m+1)D})$$

$$V(\boldsymbol{x}_{mD}) \geq \left(1 + \frac{c_3}{c_2}\right) V(\boldsymbol{x}_{(m+1)D})$$

$$V(\boldsymbol{x}_{(m+1)D}) \leq \left(1 + \frac{c_3}{c_2}\right)^{-1} V(\boldsymbol{x}_{mD}) \leq \left(1 + \frac{c_3}{c_2}\right)^{-(m+1)} V(\boldsymbol{x}_0)$$

and (D 13) is valid for $m + 1$. If we let $k = 0$ in (73)

$$V(\boldsymbol{x}_0) \geq \left(1 + \frac{c_3}{c_2}\right) V(\boldsymbol{x}_D)$$

$$V(\boldsymbol{x}_D) \leq \left(1 + \frac{c_3}{c_2}\right)^{-1} V(\boldsymbol{x}_0)$$

we see that (D 13) is valid for $m = 1$. Therefore (D 13) is valid for all $m \geq 0$ ((D 13) is trivially satisfied for $m = 0$). Equation (D 13) and condition (i) imply that for all k such that $(m-1)D \leq k \leq mD$

$$V(\boldsymbol{x}_k) \leq \left(1 + \frac{c_3}{c_2}\right)^{-(m-1)} V(\boldsymbol{x}_0)$$

Because $(k/D) - 1 \leq m - 1$ for all k such that $k \leq mD$, it follows from the above equation that

$$V(\boldsymbol{x}_k) \leq \left(\frac{1}{1 + \frac{c_3}{c_2}}\right)^{(k/D)-1} V(\boldsymbol{x}_0)$$

$$V(\boldsymbol{x}_k) \leq \left(1 + \frac{c_3}{c_2}\right) \left[\left(\frac{1}{1 + \frac{c_3}{c_2}}\right)^{1/D}\right]^k V(\boldsymbol{x}_0) \tag{D 14}$$

From (D 14) and condition (ii), we see that

$$c_1\rho(\boldsymbol{x}_k, \mathscr{X}_m) \leq \left(1 + \frac{c_3}{c_2}\right)\beta^k V(\boldsymbol{x}_0)$$
$$\leq c_2\left(1 + \frac{c_3}{c_2}\right)\beta^k \rho(\boldsymbol{x}_0, \mathscr{X}_m)$$

where $\beta = (1 + c_3/c_2)^{-1/D} < 1$. Therefore, there is some $\alpha > 0$ such that $e^{-\alpha} \geq \beta$ and

$$\rho(\boldsymbol{x}_k, \mathscr{X}_m) \leq \frac{c_2}{c_1}\left(1 + \frac{c_3}{c_2}\right) e^{-\alpha k} \rho(\boldsymbol{x}_0, \mathscr{X}_m). \quad \square$$

Clearly, (D 11) satisfies condition (iii) of Lemma 4. Conditions (i) and (ii) of Lemma 4 are satisfied in the proof of stability. □

References

Boel, R., and Schuppen, J. van, 1989, Distributed routing for load balancing. *Proceedings of the Institute of Electrical and Electronics Engineers*, pp. 210–221.

Burgess, K., 1992, Stability and boundedness analysis of discrete event systems. Master's thesis, Ohio State University, Department of Electrical Engineering.

Cybenko, G., (1994), Dynamic load balancing for distributed memory multi-processors. *Journal of Parallel and Distributed Computing*, to be published.

Michel, A., Wang, K., and Passino, K. M., 1992 a, Comparison theory for general motions of dynamical systems with applications to discrete event systems. In *Proceedings of the International Symposium on Circuits and Systems*, San Diego, California; 1992 b, On stability preserving mappings of dynamical systems—Part I: Theory; Part II: Applications, *International Journal of Automation and Remote Control*, 1995, to be published.

Passino, K. M., and Antsaklis, P. J., 1993, Modeling and analysis of artificially intelligent planning systems. *An Introduction to Intelligent and Autonomous Control* (Massachussetts: Kluwer Academic Press), Chap. 8.

Tsitsiklis, J., and Bertsekas, D., 1989, *Parallel and Distributed Computation* (Englewood Cliffs, NJ: Prentice-Hall).

Passino, K. M., Michel, A., and Antsaklis, P., 1990, Stability analysis of discrete event systems. *Proceedings of the 28th Allerton Conference on Communication, Control, and Computing*, University of Illinois, Urbana-Champaign, pp. 487–496; 1991, Lyapunov stability of a class of discrete event systems. *Proceedings of the American Control Conference*, pp. 2911–2916; 1994, Lyapunov stability of a class of discrete event systems. *IEEE Transactions on Automatic Control*, **39**(2), 269–279.

Shivaratri, N., Krueger, P., and Singhal, M., 1992, Load distributing for locally distributed systems. *Computer*, **25**, 33–44.