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A fuzzy dynamic model based state estimator

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Abstract

Systems containing uncertainty are traditionally analyzed with probabilistic methods. However, for non-linear, non-Gaussian systems solutions can sometimes be very difficult to obtain. The focus of this work is to determine if in such cases fuzzy dynamic system models may provide an alternative approach that more easily leads us to a good solution. In this paper, we present a fuzzy estimator whose system model is a fuzzy dynamic system. We show that for the linear, Gaussian case the fuzzy estimator produces the same result as the Kalman filter. More importantly, we show that the fuzzy estimator can succeed for some non-Gaussian, nonlinear systems. Finally, we illustrate the application of the fuzzy estimator on a non-linear, non-Gaussian, time-varying rocket launch problem where we show that it performs better than the extended Kalman filter. From a broad perspective this paper essentially shows how to build on Zadeh's seminal ideas in fuzzy sets, logic, and systems and use Kalman's seminal ideas on optimal estimators to construct a novel fuzzy estimator for non-linear estimation problems. While this seems to reconcile some of the fundamental ideas of Zadeh and Kalman it is unfortunate that the fuzzy estimator can be very computationally complex to implement for practical applications.

Keywords: Fuzzy dynamic systems; State estimation; Probability theory and statistics; Engineering

1. Introduction

In control system and estimator design, the establishment of a system model is quite fundamental. Many traditional approaches assume that models are available in the form of ordinary differential equations or discrete-time difference equations. In the cases where there exists uncertainty the models are generally assumed to be in the form of stochastic differential or difference equations. In many cases it is very difficult to obtain stochastic models, especially if the process is highly complex. Even when models are available it is sometimes difficult to propagate the uncertainty through the system dynamics according to the axioms of probability theory, especially for non-linear systems. Therefore, in this article, we examine the use of an alternative yet practical framework of fuzzy dynamic system models for dealing with systems exhibiting uncertain behavior. It is hoped that the use of these fuzzy dynamic models will provide a foundation on which to build solutions to

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challenging engineering problems that would otherwise be difficult to solve using probabilistic methods.

This article introduces an "optimal" state estimator, based on fuzzy set theory, that is capable of dealing with systems with random disturbances and "uncertainty". We will refer to this as a fuzzy dynamic model based state estimator (or "fuzzy estimator" for short) as we will use a fuzzy dynamic system model of the process in the estimator. As one would imagine, this estimator should be similar to that of the Kalman filter [15] when a linear system with Gaussian disturbances is considered. To illustrate these similarities, we show here that the fuzzy estimator is, in a sense to be made clear later, equivalent to the Kalman filter under these conditions. While such a result may not be surprising to persons familiar with conventional estimation theory, it is of significant value to the fuzzy systems community as it draws connections to conventional ideas and helps provide insights into the behavior of general fuzzy dynamic systems. Further, the fuzzy estimator is similar to the nonlinear filtering approach based on the Fokker–Planck equation presented in [25].

The fuzzy estimator introduced in this article relies on a fuzzy dynamic system model of the process and the uncertainty. The fuzzy dynamic system model is used to propagate uncertainty in a way analogous to that of the mean and covariance estimator of a Kalman filter. Here, the membership function is viewed as being analogous to a probability density function. In this way, the fuzzy dynamic system model lends itself directly for use in a state estimator for non-linear, non-Gaussian systems. Note however, this is done with significant increase in computational complexity as the fuzzy dynamic system model is propagating a function and not just two parameters of a function like mean and covariance. Next, we overview related research.

Over recent years, fuzzy dynamic system models have been used in several control applications. For example, Chand and Hansen [2] developed a fuzzy roll controller for a flexible aircraft wing. Cumani [3] develops analysis techniques for fuzzy feedback systems where both the plant and the controller are modeled as fuzzy systems. Oliveira and Lemos [18] use fuzzy dynamic models for long-range predictive control. There a control law was generated to minimize a quadratic cost. In [17] Oliveira examined the optimal linguistic/numeric interface that exists in many control systems employing fuzzy dynamic systems. Pedrycz and Oliveira [22] examine dynamic models using static fuzzy systems where dynamics enter the model by tapped delay lines. It is shown that the internal feedback can generate a null fuzzy representation of the state in the absence of fuzzy sets at the inputs of the static model. Therefore, they advocate the use of a pretuned fuzzy model input–output interface as a "regenerative device".

Others have studied identification of fuzzy dynamic systems (i.e. the use of algorithms and plant data to generate fuzzy dynamic models). Each of the system identification methods described next are based on finding the fuzzy relation R in the fuzzy dynamic system ($X_{k+1} = X_k \circ R$). [1] For example Lee et al. [13] developed a two-stage method for fuzzy dynamic system identification. The first stage involves developing a heuristic approach to approximate the fuzzy relation. The second stage is a recursive identification algorithm based on the prediction error and minimization of a quadratic performance index. Pedrycz in [20] gives a general discussion on fuzzy identification, prediction, sensitivity, and stability. Finally, Shaw and Kruger [23] present another recursive estimator. They compare the results of both an identified first- and second-order fuzzy model with a set of industrial data.

Some have examined adaptive control via simultaneous system identification and control of fuzzy dynamic systems. An example of this is given by Czogala and Pedrycz [4,5]. Furthermore, Pedrycz et al. [21] provide a short paper expanding on some of the ideas concerning identification given in [4]. In [7], Graham and Newell perform fuzzy adaptive control of a simple first-order process. The same technique was used by Graham and Newell [6] to perform fuzzy identification and control of a liquid level rig.

Although fuzzy set theory has been used in other ways in state estimators [16,24] and one can, of course, use least squares, gradient, and clustering methods to train standard and Takagi–Sugeno fuzzy systems to perform an estimation task [19], it seems that little attention has been paid to the application of fuzzy dynamic system models in estimators. This is the focus of this paper an early version of which appeared in [10].

[1] This notation will be explained in detail in Section 2.

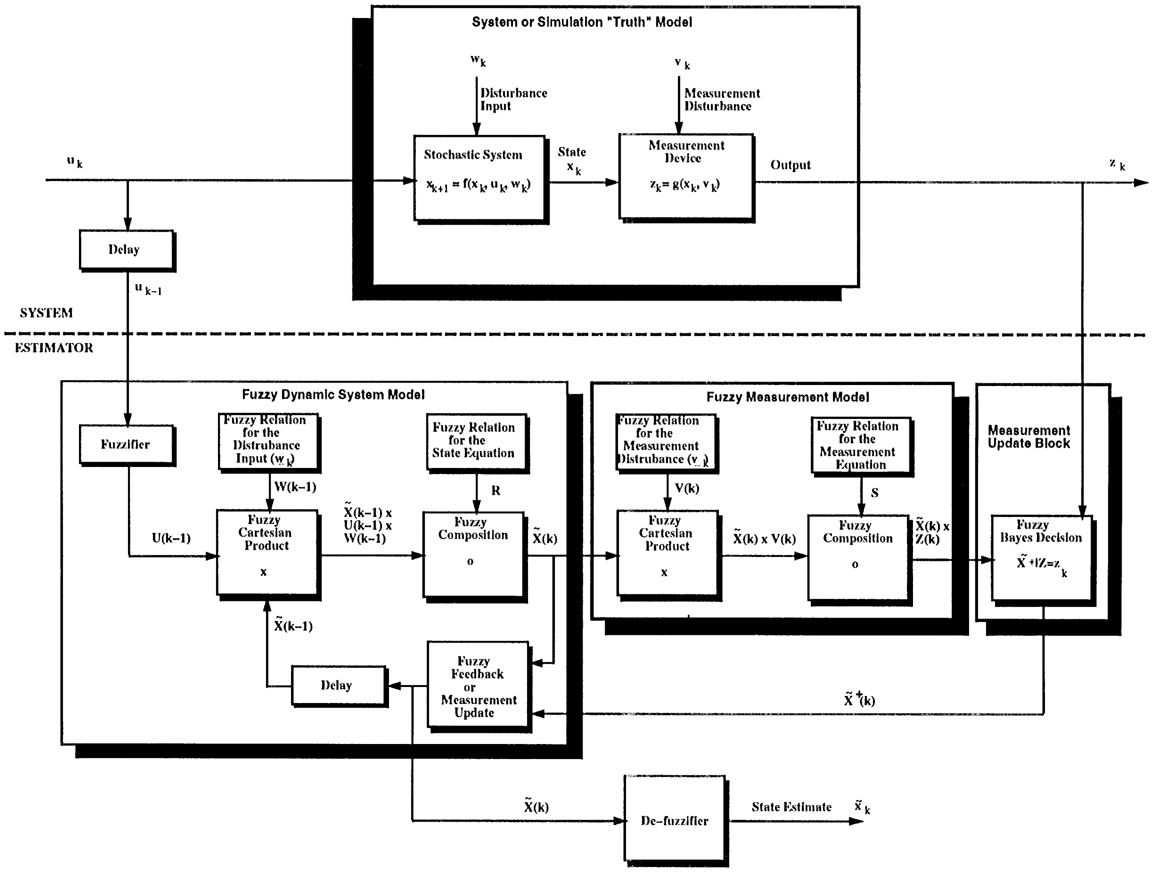

Fig. 1. Block diagram of the fuzzy model-based state estimator.

2. Basic framework

The basic framework of the fuzzy dynamic model based state estimator is shown in Fig. 1. The process whose states are to be estimated includes the system and the output measuring device, each with corresponding disturbances. As shown in Fig. 1, the system's dynamics are expressed by a discrete-time stochastic equation of the form

$$x_{k+1} = f(x_k, u_k, w_k), \tag{1}$$

$$z_k = g(x_k, v_k), \tag{2}$$

where $x_k \in \mathbb{R}^n$ is the system state, $u_k \in \mathbb{R}^r$ is the system input, $w_k \in \mathbb{R}^p$ is a *white-noise* input disturbance, $z_k \in \mathbb{R}^s$ is the process output, and $v_k \in \mathbb{R}^l$ is a white-noise output disturbance. In general, $f : \mathbb{R}^{n+r+p} \to \mathbb{R}^n$ and $g : \mathbb{R}^{n+l} \to \mathbb{R}^s$ are non-linear functions. We shall refer to this model as the process "truth model" as it should be the best known representation of the real system behavior. Notice in the above system model that the system input is assumed to be measured perfectly. We do this without loss of generality since the "disturbances" or errors in measuring the process input u_k can be absorbed into w_k. Also, it is not necessary that measurements are available at every time step kT. In many applications the measurements may become unavailable for periods of time or they may occur at a varying rate over time.

The fuzzy estimator consists of three basic components, namely a fuzzy dynamic system model, a fuzzy measurement model, and a measurement update block; hence, conceptually its structure bears some similarity to the Kalman filter [15]. A detailed description of each of these components is presented next.

2.1. Fuzzy dynamic system model

The fuzzy dynamic system model is designed so that it accurately characterizes the system behavior and its uncertainty. As illustrated in Fig. 1, it measures the system input and predicts the true system output and uncertainty in the form of a fuzzy relational state variable. Notice that unlike what has been presented in the current literature, the fuzzy dynamic system modeled here is propagating the full fuzzy relation describing the state and not a fuzzy set for each independent state variable. The reason for doing this is due to the fact that the fuzzy relation describing the state vector has a membership function that can be considered analogous to a joint probability density function. By propagating the full fuzzy relation we capture and maintain information similar to the cross-correlation. Those who have worked extensively with the Kalman filter will appreciate the benefits of having good cross-correlation information. Ignoring it is like having all the off-diagonal elements of the Kalman filter covariance matrix forced to zero. Next we explain how to produce the fuzzy dynamic system model in Fig. 1 from the stochastic system model above.

Using the (non-fuzzy) stochastic model in Eq. (1) or a reduced-order approximation of this model, the fuzzy dynamic system model in Fig. 1 can be generated using fuzzy composition or by use of the fuzzy extension principle. The details of how this is done is described later in this section. First, we will review some basic fuzzy system theory to establish notation for this paper. We begin by defining the triangular norm and co-norm [9].

Triangular norm: The triangular norm, denoted $\otimes : [0,1]\times,\dots,\times[0,1]\rightarrow[0,1]$, is an operation defined for n membership function values $\{x_1,x_2,\dots,x_n\}$. Commonly used triangular norms in fuzzy system design include the following:

$$\otimes\{x_1,x_2,\dots,x_n\} = \min\{x_1,x_2,\dots,x_n\}, \tag{3}$$

$$\otimes\{x_1,x_2,\dots,x_n\} = x_1\cdot x_2\cdot,\dots,\cdot x_n, \tag{4}$$

$$\otimes\{x_1,x_2,\dots,x_n\} = \max\{0, x_1+x_2+,\dots,+x_n-n+1\}. \tag{5}$$

Triangular co-norm: The triangular co-norm, denoted $\oplus : [0,1]\times,\dots,\times[0,1]\rightarrow[0,1]$, is an operation defined for n membership function values $\{x_1,x_2,\dots,x_n\}$.

Commonly used triangular co-norms in fuzzy system design include the following:

$$\oplus\{x_1,x_2,\dots,x_n\} = \max\{x_1,x_2,\dots,x_n\}, \tag{6}$$

$$\begin{aligned}\oplus\{x_1,x_2,\dots,x_n\} &= x_1+x_2+,\dots,+x_n-x_1x_2-x_1x_3-,\dots,-x_{n-1}x_n\\ &\quad +x_1x_2x_3+,\dots,+x_{n-2}x_{n-1}x_n-,\dots,\\ &\quad +/-x_1x_2,\dots,x_n,\end{aligned} \tag{7}$$

$$\oplus\{x_1,x_2,\dots,x_n\} = \max\{1,x_1,x_2,\dots,x_n\}. \tag{8}$$

It is important to point out that the name "triangular norm" is used since all properties of a norm are applicable over the domain [0, 1].

Using definitions of the triangular norm and co-norm the operations of union and intersection of ordinary sets can be extended to fuzzy sets. Consider the fuzzy sets U^1 and U^2 defined on the universe of discourse $\mathscr{U}$ with membership functions $\mu_{U^1}(u)$ and $\mu_{U^2}(u)$, respectively. The most widely accepted definitions of fuzzy set operations are as follows.

Union: The union of U^1 and U^2, denoted $U^1\cup U^2$, is a fuzzy set with a membership function defined by

$$\mu_{U^1\cup U^2}(u) = \oplus\{\mu_{U^1}(u), \mu_{U^2}(u) \colon u\in\mathscr{U}\}. \tag{9}$$

Intersection: The intersection of U^1 and U^2, denoted $U^1\cap U^2$, is a fuzzy set with a membership function defined by

$$\mu_{U^1\cap U^2}(u) = \otimes\{\mu_{U^1}(u), \mu_{U^2}(u) \colon u\in\mathscr{U}\}. \tag{10}$$

The $\otimes$ and $\oplus$ symbols are used to denote any triangular norm or co-norm, respectively. Now consider fuzzy set operations over more than one universe of discourse such as the Cartesian product.

Cartesian product: If $U_1, U_2,\dots,U_r$ are fuzzy sets defined as subsets of the universes of discourse $\mathscr{U}_1,\mathscr{U}_2,\dots,\mathscr{U}_r$, respectively, the Cartesian product is a fuzzy set, denoted $U_1\times U_2\times\cdots\times U_r$, with a membership function defined by

$$\mu_{U_1\times U_2\times\cdots\times U_r}(u_1,u_2,\dots,u_r) = \otimes\{\mu_{U_1}(u_1), \mu_{U_2}(u_2),\dots,\mu_{U_r}(u_r)\}, \tag{11}$$

where $u_1, u_2, \ldots, u_r$ are elements of $\mathscr{U}_1, \mathscr{U}_2, \ldots, \mathscr{U}_r$ respectively.

The Cartesian product of two or more fuzzy sets is called a *fuzzy relation*. Next, we can define *fuzzy composition* and the *fuzzy extension* principle. These operations will be used later in this article to determine functions of fuzzy sets.

Fuzzy composition: Assume that U and R are fuzzy relations defined on $\mathscr{U}_1 \times \mathscr{U}_2 \times, \ldots, \times \mathscr{U}_r$ and $\mathscr{U}_1 \times \mathscr{U}_2 \times, \ldots, \times \mathscr{U}_r \times \mathscr{Y}$, respectively, the "fuzzy composition" of U and R is a fuzzy relation denoted by $U \circ R$ and has a membership function defined as

$$\begin{aligned}\mu_{U \circ R}(y) &= \oplus\{\otimes\{\mu_U(u_1, u_2, \ldots, u_r),\\ &\quad \mu_R(u_1, u_2, \ldots, u_r, y): (u_1, u_2, \ldots, u_r)\\ &\quad \in \mathscr{U}_1 \times \mathscr{U}_2 \times, \ldots, \times \mathscr{U}_r \text{ and } y \in \mathscr{Y}\}:\\ &\quad (u_1, u_2, \ldots, u_r) \in \mathscr{U}_1 \times \mathscr{U}_2 \times, \ldots, \times \mathscr{U}_r\}. \qquad (12)\end{aligned}$$

Fuzzy extension principle: Let U be a fuzzy set defined on the universe $\mathscr{U} = \mathscr{U}_1 \times \mathscr{U}_2 \times, \ldots, \times \mathscr{U}_r$ with membership function $\mu_U(u_1, u_2, \ldots, u_r)$. Let T be a mapping from $\mathscr{U}_1 \times \mathscr{U}_2 \times, \ldots, \times \mathscr{U}_r$ to $\mathscr{Y}_1 \times \mathscr{Y}_2 \times, \ldots, \times \mathscr{Y}_s$ such that $y = T(u_1, \ldots, u_r)$. The extension principle states that fuzzy set $Y = T(U)$ has a membership function given by

$$\mu_Y(y) = \begin{cases} \oplus\{\mu_U(u_1, \ldots, u_r): u_1, u_2, \ldots, u_r, \\ \quad y = T(u_1, \ldots, u_r)\} \\ \mu_Y(y) = 0 \quad \text{if } T^{-1}(y) = \emptyset \end{cases} \qquad (13)$$

where $T^{-1}(y)$ is the inverse image of y and $\emptyset$ denotes the null set.

The fact that a triangular norm is used in the definition of the fuzzy composition and extension principle is important. The *triangular inequality* ensures that we are no more certain about the "outputs" (y) than we are about the inputs (u). Hence the basic mechanism behind the mapping of fuzzy sets makes some intuitive sense. Now that we have defined notation we now consider the construction fuzzy dynamic systems.

2.1.1. Construction of fuzzy dynamic model via fuzzy composition

To use fuzzy composition it is convenient to generate a list of fuzzy rules that characterizes the behavior of the stochastic dynamic behavior in Eq. (1). A rule base is generated so that it contains enough rules to ensure *completeness* (i.e. there exist enough rules that at least one is activated for all realizations of the vectors x_k, u_k, and w_k). Suppose the ith rule is defined so that it is activated in the region of space around the set of values $x_k = x^i \in \mathbb{R}^n$, $u_k = u^i \in \mathbb{R}^r$, and $w_k = w^i \in \mathbb{R}^p$. Here we define the following fuzzy relations:

$$X_k^i = \mathscr{F}(x^i), \qquad (14)$$

$$U^i = \mathscr{F}(u^i), \qquad (15)$$

$$W^i = \mathscr{F}(w^i), \qquad (16)$$

$$X_{k+1}^i = \mathscr{F}(f(x^i, u^i, w^i)), \qquad (17)$$

$$U(k) = \mathscr{F}(u_k), \qquad (18)$$

where $\mathscr{F}$ represents the process of fuzzification around a single element. The subscripts on X_k^i and X_{k+1}^i are used only to distinguish cases; not to indicate that they are changing in time.

To illustrate the fuzzification operator $\mathscr{F}$, consider the vector $x^i = [x_1^i, \ldots, x_n^i]^{\mathrm{T}}$. One possible fuzzification scheme might produce a fuzzy singleton relation $X^i = \mathscr{F}(x^i)$ with membership function

$$\mu_{X^i}(x) = \begin{cases} 1 & \text{if } \underline{x} = \underline{x}^i, \\ 0 & \text{otherwise.} \end{cases} \qquad (19)$$

Or another possibility might be a normalized jointly Gaussian function

$$\mu_{X^i}(x) = \mathrm{e}^{(1/2)[(x - x^i)^{\mathrm{T}} C^{-1} (x - x^i)]}, \qquad (20)$$

where C is a symmetric positive-definite matrix. Hence the fuzzification operator simply turns crisp values into fuzzy sets.

Using the fuzzy relations defined in Eqs. (14)–(18), one can generate a fuzzy knowledge base such that the ith rule is given by

If $X(k)$ is X_k^i **and** $U(k)$ is U^i **and** $W(k)$ is W^i
Then $X(k+1)$ is X_{k+1}^i

where $X(k)$ and $X(k+1)$ are fuzzy relations characterizing the current and the next state, respectively, $W(k)$ is a fuzzy relation characterizing the process disturbance input, and $U(k)$ is a fuzzy relation characterizing the process input.

This rule can be denoted mathematically by a fuzzy relation of the form

$$R^i = X_k^i \times U^i \times W^i \times X_{k+1}^i. \tag{21}$$

Assuming there exist many such rules, the overall fuzzy relation describing the entire rule base is denoted by the union of these rules

$$R = \bigcup_i R^i. \tag{22}$$

Thus, the fuzzy dynamic system model can be expressed by the following:

$$X(k+1) = [X(k) \times U(k) \times W(k)] \circ R. \tag{23}$$

The choice of the fuzzy relation $W(k)$ is based on a fuzzy characterization of the process disturbance at time kT. One possible choice for the membership function of $W(k)$ is simply a normalized version of the joint probability density function of w_k. In many cases, the characteristics of the process disturbance input will not change in time so often the fuzzy relation $W(k)$ will be constant.

2.1.2. *Construction of fuzzy dynamic model via the fuzzy extension principle*

Another way to construct a fuzzy dynamic model is by transforming the stochastic model in Eqs. (1) and (2) to a fuzzy model via the fuzzy extension principle. First consider Eq. (1). For time kT define $X(k)$ to be the fuzzy relation describing the state, $U(k)$ is the fuzzy relation for the process input, and $W(k)$ is the fuzzy relation for the process input disturbance. Hence, the fuzzy relation for the process state at time $(k+1)T$ can be found via the fuzzy extension principle

$$\mu_{X_{k+1}}(x_{k+1}) = \begin{cases} \oplus\{\mu_{X(k)\times U(k)\times W(k)}(x_k, u_k, w_k): \\ \quad x_k, u_k, w_k, x_{k+1} = f(x_k, u_k, w_k)\} \\ 0 \quad \text{if } f^{-1}(x_{k+1}) = \emptyset. \end{cases} \tag{24}$$

This is the approach we will use in the development of the fuzzy estimator.

2.2. *Measurement update block*

In the fuzzy estimator in Fig. 1 measurement updates are performed using conditional fuzzy sets. As will be shown, this method of performing measurement updates places certain constraints on the design of the fuzzy measurement model. Consequently, it is difficult to explain the design of the fuzzy measurement model without first describing the operation of the measurement update block.

The measurement update block performs the function of improving the fuzzy relational state estimate $X(k)$ using information provided by system measurements. Here we derive the fuzzy measurement update equations in a "fuzzy Bayesian" manner by generating a new fuzzy relation for the state conditioned on system measurements.

Since the state equation in Eq. (1) is Markov, the algorithm will be recursive. If the process is Markovian, measurement updates need only be based on the current measurement since by definition previous measurements will not provide additional information. However, if the process is non-Markovian, some benefit may be gained by performing updates based on some portion of the measurement history. However, the determination of what portion of the measurement history should be used is beyond the scope of this article.

A fuzzy set, denoted A, conditioned on fuzzy set $B = \bar{b}$ (a fuzzy singleton), denoted $A \mid B = \bar{b}$ where $\bar{b}$ is a specific element on the universe of discourse for B, has a membership function often defined by the following:

$$\mu_{A|B=\bar{b}}(a) = \frac{\mu_{A\times B}(a, \bar{b})}{\mu_B(\bar{b})}, \tag{25}$$

where $\mu_{A\times B}(a, b)$ is the membership function for $A \times B$ and $\mu_B(b)$ is the membership function for B (this is a "fuzzy Bayesian rule"). Note that since the denominator of Eq. (25) is a scalar constant for a specific value of $b = \bar{b}$, it acts only as a scaling factor. Therefore, for reasons that will become clear in later sections, we propose another definition for the conditional fuzzy set

$$\mu_{A|B=\bar{b}}(a) = \mathcal{N}(\mu_{A\times B}(a, \bar{b})), \tag{26}$$

where $\mathcal{N}$ is a normalization operation that normalizes the functions so that peak has a value of 1. Note that this definition only requires $A \times B$ and not B. Consequently, use of this definition results in a reduction of the number of computations required by the fuzzy measurement model.

Now that we have defined conditional fuzziness, it is possible to derive the measurement update equation. Since the process in Eqs. (1) and (2) is Markov, measurement updates are to be performed based on the current measurements. The fuzzy relation, $\tilde{X}^{+}(k)$, describing the fuzzy state after the measurement update is given by

$$\tilde{X}^{+}(k)=\tilde{X}(k) \mid Z(k)=z_k \tag{27}$$

which has a membership function

$$\begin{aligned}\mu_{\tilde{X}+(k)}(x_k) &= \mu_{\tilde{X}(k)|Z(k)=z_k}(x_k,z_k)\\ &= \mathcal{N}(\mu_{\tilde{X}(k)\times Z(k)}(x_k,z_k)).\end{aligned} \tag{28}$$

Hence, the fuzzy measurement model described next should be designed so that it computes an estimate of $\tilde{X}(k)\times Z(k)$.

2.3. Fuzzy measurement model

The fuzzy measurement model is designed so that it properly characterizes the measuring device and its white noise disturbances v_k. It receives the fuzzy state relation estimate $\tilde{X}(k)$ from the fuzzy dynamic system model to generate a prediction of the fuzzy relation $\tilde{X}(k)\times Z(k)$. Similar to the fuzzy dynamic system model, the measurement model is designed using the fuzzy extension principle or fuzzy composition. For example, if $\mu_{\tilde{X}(k)}(x_k)$ and $\mu_{V(k)}(v_k)$ are the membership functions for fuzzy sets $\tilde{X}(k)$ and $V(k)$, respectively, their Cartesian product can be formed to obtain $\mu_{\tilde{X}(k)\times V(k)}(x_k,v_k)$. Using the fuzzy extension principle we get

$$\mu_{\tilde{X}(k)\times Z(k)}(x_k,z_k) = \begin{cases} \oplus\{\mu_{\tilde{X}(k)\times V(k)}(x_k,v_k) : z_k=g(x_k,v_k)\} \\ 0 \quad \text{if there does not exist } v_k \text{ such that} \\ \qquad z_k=g(x_k,v_k) \text{ is satisfied.}\end{cases}$$

It is sometimes desirable to design the fuzzy dynamic system model based on a reduced order approximation of the stochastic model in Eq. (1). As a result of this approximation, added uncertainty is created. Therefore, sometimes it becomes necessary to absorb this added uncertainty into the characterization of $V(k)$ (and possibly $W(k)$). Of course, this opens a whole set of issues concerning reduced-order design optimization which is beyond the scope of this article.

2.4. Defuzzification and optimality

So far, we have described a method for generating a fuzzy estimate of the system state using a fuzzy model of the process and measurement feedback based on conditional fuzzy sets. However, in many applications the fuzzy relation estimate $\tilde{X}(k)$ must be defuzzified to a specific vector $\tilde{x}(k)$ to be of practical use. Hence, the optimality of the $\tilde{x}(k)$ depends greatly on the chosen defuzzification method.

In the next sections of this article, we show that when the system is linear, the disturbances are Gaussian, and the fuzzy estimator is designed as specified above the membership function for $\tilde{X}(k)$ is a "normalized version" of the joint probability density function that would be obtained using a Kalman filter. Although it is yet to be proven, there is no reason to doubt that the fuzzy estimator might produce a "good" estimate of the normalized joint probability density function (membership function) for some non-linear systems with non-Gaussian disturbances that are characterized with fuzzy sets. When it is possible to show that the fuzzy estimator produces a good estimate of the normalized joint probability function, one can talk about optimality using probabilistic arguments similar to those used for the Kalman filter. Often, analysis of this type will provide insight to the best possible scheme for defuzzification.

In the Kalman filter, the conditional mean is chosen as the optimal estimate because it is the mean, mode, and median of the joint Gaussian probability density function. When the membership function of the state relation estimate $\tilde{X}(k)$ is considered analogous to a joint probability density function, computation of the mean is akin to performing a *center of area* defuzzification [11,12] (or *center of volume*, etc. for higher-dimension relations). Likewise, computation of the mode is akin to performing a *mean of maximum* defuzzification [11,12]. Hence, in general, the defuzzification algorithm will be either the center of area or mean of maximum. Maybeck [15] describes a number of optimality criterion used for specifying techniques for extracting the optimal solution from a probability density function. These might actually assist in the choice of defuzzification strategies.

3. Linear Gaussian case

In this section we consider linear systems with Gaussian-shaped membership functions. We will show that the fuzzy estimator described above will produce a result that is analogous to Kalman filter in the sense that the "center" and "spread" of the membership functions propagate the same as the mean and covariance of the Kalman filter.

3.1. Truth model

Here the truth model is a linear stochastic model of the form

$$x_{k+1} = Ax_k + w_k, \tag{29}$$

$$z_k = Hx_k + v_k, \tag{30}$$

where $x_k, x_{k+1} \in \mathbb{R}^n$ are the current and next state, respectively, $w_k \in \mathbb{R}^p$ is a white noise process disturbance, $z_k \in \mathbb{R}^m$ is the current measurement, $v_k \in \mathbb{R}^l$ is a white noise output measurement disturbance, and $A \in \mathbb{R}^{n\times n}$, $H \in \mathbb{R}^{m\times n}$ are constant matrices. Notice that to simplify the math below we set the process input $u_k \in \mathbb{R}^r$ to zero. Also, to obtain the result presented in this section, A is assumed to be invertible which implies that A^{T} is invertible (i.e. the system has no eigenvalues at zero). Provided that the above linear discrete-time model was created from a continuous-time system A will always be invertible so this does not present a serious restriction.

Let the uncertainty in x_k, w_k, and v_k be modeled by Gaussian-shaped membership functions of the form

$$\mu_{\tilde{X}(k)}(x_k) = \exp\left\{-\tfrac{1}{2}[(x_k - \tilde{x}_k)^{\mathrm{T}} P_k^{-1}(x_k - \tilde{x}_k)]\right\}, \tag{31}$$

$$\mu_{W(k)}(w_k) = \exp\left\{-\tfrac{1}{2}[(w_k)^{\mathrm{T}} Q^{-1}(w_k)]\right\}, \tag{32}$$

$$\mu_{V(k)}(v_k) = \exp\left\{-\tfrac{1}{2}[(v_k)^{\mathrm{T}} R^{-1}(v_k)]\right\}, \tag{33}$$

where $\tilde{x}_k$ denotes the "center" for the state, $P_k \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{p\times p}$, $R \in \mathbb{R}^{l\times l}$ denote the "spread" in the state, input disturbance, and output disturbance, respectively.

3.2. Propagation of the state

If we use the product for the triangular norm in the Cartesian product, we get

$$\mu_{\tilde{X}(k)\times W(k)}(x_k, w_k) = \exp\{-\tfrac{1}{2}[(x_k - \tilde{x}_k)^{\mathrm{T}} P_k^{-1}(x_k - \tilde{x}_k)] - \tfrac{1}{2}[(w_k)^{\mathrm{T}} Q^{-1}(w_k)]\}. \tag{34}$$

Hence, if we use the max operator for the triangular co-norm in the fuzzy extension principle the problem of finding $\mu_{X(k+1)}(x_{k+1})$ becomes one of maximizing Eq. (34) subject to Eq. (29). Upon performing this operation we obtain the following theorem.

Theorem 3.1. *Given a linear system as in Eq.* (29) *assume that the fuzzy extension principle was used* (*with product as the triangular norm and max for the co-norm*) *to generate* $\tilde{X}(k+1)$*. Let* $\tilde{X}(k)$ *and* $W(k)$ *have membership functions given in the Eqs.* (31) *and* (32), *respectively. The membership function for* $\tilde{X}(k+1)$ *can be shown to be the following Gaussian-shaped function*:

$$\mu_{\tilde{X}(k+1)}(x_{k+1}) = \exp\left\{-\tfrac{1}{2}[(x_{k+1} - \tilde{x}_{k+1})^{\mathrm{T}} P_{k+1}^{-1}(x_{k+1} - \tilde{x}_{k+1})]\right\}, \tag{35}$$

where

$$\tilde{x}_{k+1} = A\tilde{x}_k, \tag{36}$$

$$P_{k+1} = A^{\mathrm{T}} P_k A + Q. \tag{37}$$

Theorem 3.1 is proved fully in Appendix A. Note that Eqs. (36) and (37) are the same as those obtained for the mean and variance in the discrete-time Kalman filter during state propagation [15]. Although these equations are the same as the Kalman filter equations, they were obtained in a very different manner, namely from axioms of fuzzy sets and not probability theory. Hence, they cannot be interpreted the same as the Kalman equations. They are update equations for membership functions for a fuzzy dynamic system. The same can be said of the update equations found in Eqs. (43) and (44) for the measurement update equations in the next section.

3.3. Measurement updates

Since the given stochastic process is known to be Markovian, measurement updates are performed using the following equation:

$$\tilde{X}^+(k) = \tilde{X}(k) \mid Z(k) = z_k \tag{38}$$

whose membership function is given by

$$\mu_{\tilde{X}^+(k)}(x_k) = \mathcal{N}(\mu_{\tilde{X}(k)\times Z(k)}(x_k, z_k)). \tag{39}$$

If we use the product as the triangular norm in a Cartesian product we get

$$\mu_{X(k)\times V(k)}(x_k, v_k) = \exp\left\{-\tfrac{1}{2}[(x_k - \tilde{x}_k)^{\mathrm{T}} P_k^{-1}(x_k - \tilde{x}_k)] - \tfrac{1}{2}[(v_k)^{\mathrm{T}} R^{-1}(v_k)]\right\}. \tag{40}$$

Hence, if we use the max operator for the triangular co-norm in the fuzzy extension principle the problem of finding $\mu_{X(k)\times Z(k)}(x_k, z_k)$ becomes one of maximizing Eq. (40) subject to Eq. (30). Then we normalize to obtain $\mu_{X^+(k)}(x_k^+)$. Upon performing these operations we get the following theorem.

Theorem 3.2. *Given a linear system as in Eq.* (30) *assume that the fuzzy extension principle* (*with product as the triangular norm and max as the co-norm*) *is used to generate* $\tilde{X}(k+1)$. *Let* $\tilde{X}(k)$ *and* $V(k)$ *have membership functions given in the Eqs.* (31) *and* (33), *respectively*, *the membership function for* $\tilde{X}^+(k)$ *can be shown to be the following Gaussian-shaped function*:

$$\mu_{\tilde{X}^+(k)}(x_k^+) = \mathcal{N}(\mu_{\tilde{X}(k)\times Z(k)}(x_k, z_k)) \tag{41}$$

$$= \exp\{-\tfrac{1}{2}[(x_k - \tilde{x}_k^+)^{\mathrm{T}}(P_k^+)^{-1}(x_k - \tilde{x}_k^+)]\}, \tag{42}$$

where

$$\tilde{x}_{k+1}^+ = P_k^+ P^{-1}\tilde{x}_k + P_k^+ H^{\mathrm{T}} R^{-1} z_k, \tag{43}$$

$$P_k^+ = [P_k^{-1} + H^{\mathrm{T}} R^{-1} H]^{-1}. \tag{44}$$

The proof of Theorem 3.2 is shown in Appendix A. Note that Eqs. (43) and (44) are the same as that which are obtained for the mean and variance in the discrete-time Kalman filter during measurement updates [15]. While for the linear Gaussian case we are able to show very close relationships between the fuzzy estimator and Kalman filter via Theorems 3.1 and 3.2, in the non-linear case the two methods produce quite different results. For the non-linear case one normally resorts to using the "extended" Kalman filter [15] which relies on the use of the system model and an on-line linearization of this model. The optimality and convergence properties of the Kalman filter are lost for the non-linear case which opens the possibility that other estimators may perform better. The fuzzy estimator does not need to perform an on-line linearization of the system model since its approach does not build on a linear estimation result like the extended Kalman filter. In the next section, we analytically show how to specify a fuzzy estimator that is in a sense optimal, and hence we show that it can out-perform the extended Kalman filter. Following this we provide a non-linear rocket launch estimation problem where the fuzzy estimator out-performs the extended Kalman filter.

4. Non-linear non-Gaussian case

Using fuzzy composition on discretized membership functions it is possible to implement a fuzzy estimator for practically any shaped membership function and any non-linear system. However, this is done with a significant increase in computational complexity. The reason is due to the fact that we are propagating all elements of the discretized membership function instead of a few parameters of the function such as the "center" and "spread" as was done for the linear Gaussian case.

The goal of this section is to identify membership functions that maintain their basic shape when propagated in a fuzzy dynamic system when the underlying dynamics are linear (such as the case of the Gaussian-shaped membership functions) or non-linear. We will show in this section that the uniform shaped membership function exhibits this property.

Since a jointly uniform probability function has not been defined (at least in the knowledge of the authors) on which we could base the definition of a membership function, we will constrain our analysis to first-order systems. Consider the case where the system dynamics are characterized by the following:

$$x_{k+1} = f(x_k) + w_k, \tag{45}$$

$$z_k = g(x_k) + v_k, \tag{46}$$

where $x_k \in \mathbb{R}$ is the system state, $w_k \in \mathbb{R}$ is a white noise input disturbance, $z_k \in \mathbb{R}$ is the process output, and $v_k \in \mathbb{R}$ is a white noise output disturbance. We assume that the functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are increasing or decreasing non-linear monotonic functions of x_k. Notice that we do not include a system input $u_k \in \mathbb{R}$. We do this with little loss of generality since the input signal can be absorbed into w_k.

As shown in Fig. 3, the membership functions for x_k, w_k, and v_k are uniform shaped such that at time k,

$$\mu_{X_k}(x_k) = u(x_k - a_x(k)) - u(x_k - b_x(k)), \tag{47}$$

$$\mu_{W_k}(w_k) = u(w_k - a_w(k)) - u(w_k - b_w(k)), \tag{48}$$

$$\mu_{V_k}(v_k) = u(v_k - a_v(k)) - u(v_k - b_v(k)), \tag{49}$$

where $u(\cdot)$ is the unit step function, $b_x(k) > a_x(k)$, $b_w(k) > a_w(k)$, and $b_v(k) > a_v(k)$. Notice that the uniform distribution is characterized by two parameters. Therefore, it is necessary to only keep track of $a_x(k)$ and $b_x(k)$ in the fuzzy estimator. We show how to do this next.

Due to the fact that we obtain *Dirac delta* functions when taking a derivative of uniform membership functions, we found it difficult to solve this problem using the Lagrange method that was used for the linear Gaussian case. However, since this is a simple first-order system, a graphical solution is easily obtained.

Notice that when the product is used for the triangular norm in the Cartesian product we obtain

$$\mu_{X_k \times W_k}(x_k, w_k) = \{u(x_k - a_x(k)) - u(x_k - b_x(k))\} \times \{u(w_k - a_w(k)) - u(w_k - b_w(k))\} \tag{50}$$

which is equal to zero everywhere except when $a_x(k) \leqslant x_k \leqslant b_x(k)$ and $a_w(k) \leqslant w_k \leqslant b_w(k)$ where $\mu_{X_k \times W_k}(x_k, w_k) = 1$. This region is illustrated by the gray shaded area in Figs. 4(a) and (b). Our goal is to solve the following problem:

$$\mu_{X_{k+1}}(x_{k+1}) = \sup\{\mu_{X_k \times W_k}(x_k, w_k) : x_{k+1} = f(x_k) + w_k\}. \tag{51}$$

Notice in Fig. 4(a) we show contour lines for the equation $\tilde{x}_{k+1} = f(x_k) + w_k$ where $f(x_k)$ is monotonic increasing and $\tilde{x}_{k+1}$ represents fixed values of x_{k+1}. It is easy to see that the contours that pass through the shaded region of Fig. 4(a) have a maximum value of one. All other contours have a maximum value of zero. Also it is easy to see that the minimum value of $\hat{x}_{k+1}$, denoted $a_x(k+1)$, and maximum value of $\hat{x}_{k+1}$, denoted $b_x(k+1)$, that passes through the shaded region in Fig. 4(a) are computed by

$$a_x(k+1) = f(a_x(k)) + a_w(k), \tag{52}$$

$$b_x(k+1) = f(b_x(k)) + b_w(k), \tag{53}$$

respectively. Likewise, Fig. 4(b) shows contour lines for the equation $\tilde{x}_{k+1} = f(x_k) + w_k$ where $f(x_k)$ is monotonic decreasing and $\tilde{x}_{k+1}$ represents fixed values of x_{k+1}. From Fig. 4(b) it is easy to see that the minimum value of $\tilde{x}_{k+1}$, denoted $a_x(k+1)$, and maximum value of $\tilde{x}_{k+1}$, denoted $b_x(k+1)$, that passes through the shaded region in Fig. 4(b) are computed by

$$a_x(k+1) = f(b_x(k)) + a_w(k), \tag{54}$$

$$b_x(k+1) = f(a_x(k)) + b_w(k), \tag{55}$$

respectively. Thus, the next state has a *uniform shaped* membership function given by

$$\mu_{X_{k+1}}(x_{k+1}) = u(x_{k+1} - a_x(k+1)) - u(x_{k+1} - b_x(k+1)). \tag{56}$$

It is easily shown by induction that during propagation the membership function for X_k is uniform shape for all k where the *end values* $a_x(k+1)$ and $b_x(k+1)$ are propagated according to Eqs. (52) and (53), respectively, or Eqs. (54) and (55), respectively. The best estimate of the state at time kT is computed using the center of gravity defuzzification method to obtain

$$\tilde{x}_k = \frac{b_x(k) + a_x(k)}{2}. \tag{57}$$

Next we solve the measurement update equation.

Notice that if the product is used for the triangular norm in the Cartesian product we obtain

$$\mu_{X_k \times V_k}(x_k, v_k) = \{u(x_k - a_x(k)) - u(x_k - b_x(k))\} \times \{u(v_k - a_v(k)) - u(v_k - b_v(k))\} \tag{58}$$

which is equal to zero everywhere except when $ax_k \leqslant x_k \leqslant bx_k$ and $av_k \leqslant v_k \leqslant bv_k$ where $\mu_{X_k\times v_k}(x_k, v_k) = 1$. Our goal is to solve the following problem:

$$\mu_{X_k\times Z_k}(x_k, z_k) = \sup\{\mu_{X_k\times V_k}(x_k, v_k):\; z_k = g(x_k) + v_k\}. \tag{59}$$

Since $g(x_k)$ is monotonic there is only one z_k corresponding to each x_k and v_k. Hence, the solution to Eq. (59) is computed by substituting $v_k = z_k - g(x_k)$ into Eq. (58) for v_k to obtain

$$\mu_{X_k\times Z_k} = \{u(x_k - ax_k) - u(x_k - bx_k)\} \times\{u(z_k - g(x_k) - av_k) - u(z_k - g(x_k) - bv_k)\}, \tag{60}$$

which is equal to zero everywhere except when $ax_k \leqslant x_k \leqslant bx_k$ and $g(x_k) + av_k \leqslant z_k \leqslant g(x_k) + bv_k$ where $\mu_{X_k\times Z_k}(x_k, z_k) = 1$. This region is illustrated by the gray shaded area in Figs. 2(a) and (b) for monotonic increasing and monotonic decreasing $g(x_k)$, respectively. The measurement update is computed by

$$\mu_{X_k^+}(x_k^+) = \mathcal{N}\{\mu_{X_k\times Z_k}(x_k, z_k)\}. \tag{61}$$

This is equivalent to taking a *slice* out of $\mu_{X_k\times Z_k}(x_k, z_k)$ at the level of z_k as shown in Figs. 2(a) and (b). Hence, for monotonic increasing $g(x_k)$ it is easy to see from Fig. 2(a) that the new $a_x(k)$ and $b_x(k)$ after measurement update is given by

$$a_x^+(k) = \max\{a_x(k), g(x_k) + b_v(k)\}, \tag{62}$$

$$b_x^+(k) = \min\{b_x(k), g(x_k) + a_v(k)\}, \tag{63}$$

respectively. Likewise, for monotonic decreasing $g(x_k)$ it is easy to see from Fig. 2(b) that the new $a_x(k)$ and $b_x(k)$ after measurement update is given by

$$a_x^+(k) = \max\{a_x(k), g(x_k) + a_v(k)\}, \tag{64}$$

$$b_x^+(k) = \min\{b_x(k), g(x_k) + b_v(k)\}, \tag{65}$$

respectively. In both cases it is important to check that $a_x^+(k)$ does not exceed $b_x(k)$ and that $b_x^+(k)$ is greater than $a_x(k)$. Again the best estimate of the state after measurement update is computed using the center of gravity defuzzification method to obtain

$$\tilde{x}_k^+ = \frac{b_x^+(k) + a_x^+(k)}{2}. \tag{66}$$

4.1. Extended Kalman filter

To establish notation, we summarize here the extended Kalman filter equations provided in [1] which will be used for comparison with the fuzzy estimator. Let the system of interest be characterized by the dynamics model and measurement model

$$\boldsymbol{x}_{k+1} = \boldsymbol{f}(\boldsymbol{x}_k, k) + \boldsymbol{w}_k, \tag{67}$$

$$\boldsymbol{z}_k = \boldsymbol{g}(\boldsymbol{x}_k, k) + \boldsymbol{v}_k, \tag{68}$$

respectively, where $\boldsymbol{x}_k \in \mathbb{R}^n$ is a random state vector at time $t = kT$ with covariance $\boldsymbol{P}_k \in \mathbb{R}^{n\times n}$, $\boldsymbol{w}_k \in \mathbb{R}^n$ is assumed to be a Gaussian white noise input disturbance vector independent of $\boldsymbol{x}_k$ with covariance $\boldsymbol{Q}_k \in \mathbb{R}^{n\times n}$, $\boldsymbol{z}_k \in \mathbb{R}^m$ is the process measurement vector, and $\boldsymbol{v}_k \in \mathbb{R}^m$ is assumed to be a Gaussian white noise measurement disturbance independent of $\boldsymbol{x}_k$ and $\boldsymbol{w}_k$ with covariance $\boldsymbol{R}_k \in \mathbb{R}^{m\times m}$.

The extended Kalman filter estimate is *propagated* forward from the sample time $t = kT$ to time $t = (k+1)T$ by the following difference equations:

$$\tilde{\boldsymbol{x}}_{k+1} = \boldsymbol{f}(\tilde{\boldsymbol{x}}_k, k), \tag{69}$$

$$\boldsymbol{P}_{k+1} = \boldsymbol{F}(\tilde{\boldsymbol{x}}_k, k)\boldsymbol{P}_k\boldsymbol{F}^{\mathrm{T}}(\tilde{\boldsymbol{x}}_k, k) + \boldsymbol{Q}_k, \tag{70}$$

where $\boldsymbol{F}(\boldsymbol{x}_k, k) \in \mathbb{R}^{n\times n}$ is the partial derivative matrix

$$\boldsymbol{F}(\tilde{\boldsymbol{x}}_k, k) \triangleq \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x}_k, k)}{\partial \boldsymbol{x}_k}\right|_{\boldsymbol{x}_k = \tilde{\boldsymbol{x}}_k}. \tag{71}$$

The extended Kalman filter *measurement update* incorporates the measurement $\boldsymbol{z}_k$ by means of the following equations:

$$\boldsymbol{K}_k = \boldsymbol{P}_k\boldsymbol{G}^{\mathrm{T}}(\tilde{\boldsymbol{x}}_k, k)\{\boldsymbol{G}(\tilde{\boldsymbol{x}}_k, k)\boldsymbol{P}_k\boldsymbol{G}^{\mathrm{T}}(\tilde{\boldsymbol{x}}_k, k) + \boldsymbol{R}_k\}^{-1}, \tag{72}$$

$$\tilde{\boldsymbol{x}}_k^+ = \tilde{\boldsymbol{x}}_k + \boldsymbol{K}_k\{\boldsymbol{z}_k - \boldsymbol{g}(\tilde{\boldsymbol{x}}_k, k)\}, \tag{73}$$

$$\boldsymbol{P}_k^+ = \boldsymbol{P}_k - \boldsymbol{K}_k\boldsymbol{G}(\tilde{\boldsymbol{x}}_k, k)\boldsymbol{P}_k, \tag{74}$$

where $\boldsymbol{G}^{\mathrm{T}}(\tilde{\boldsymbol{x}}_k, k) \in \mathbb{R}^{m\times n}$ is the partial derivative matrix

$$\boldsymbol{G}(\tilde{\boldsymbol{x}}_k, k) \triangleq \left.\frac{\partial \boldsymbol{g}(\boldsymbol{x}_k, k)}{\partial \boldsymbol{x}_k}\right|_{\boldsymbol{x}_k = \tilde{\boldsymbol{x}}_k}. \tag{75}$$

For more details see [15].

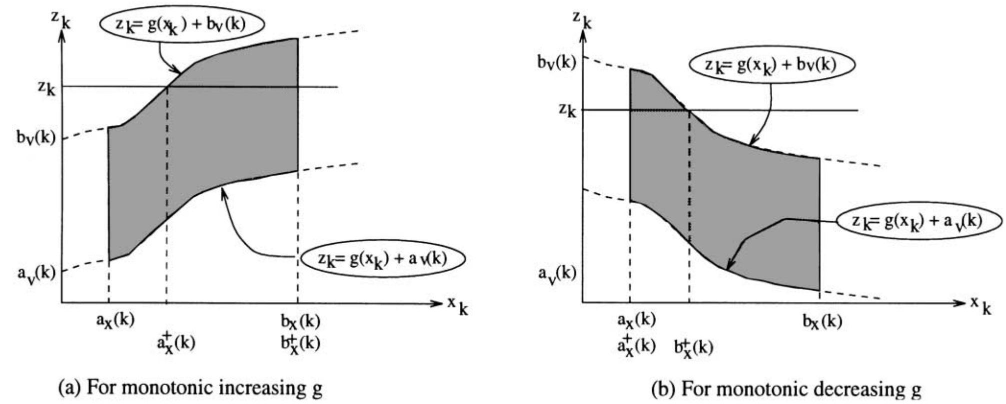

Fig. 2. Uniform membership functions for $X_k \times Z_k$ with example contour lines for (a) a monotonic increasing function $g(x_k)$ and for (b) a monotonic decreasing function $g(x_k)$.

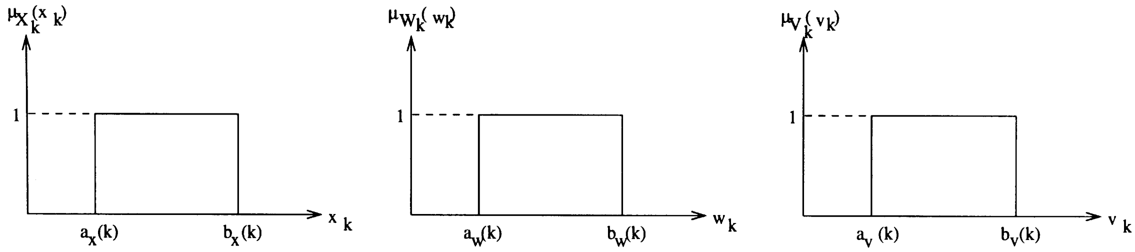

Fig. 3. Uniform membership functions for x_k, w_k, and v_k.

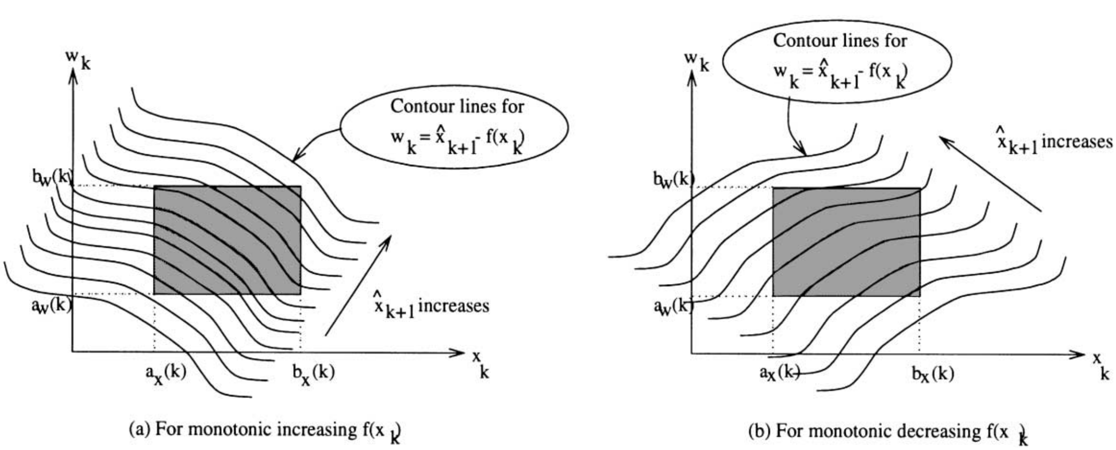

Fig. 4. Uniform membership functions for $X_k \times W_k$ with example contour lines for (a) a monotonic increasing function $f(x_k)$ and for (b) a monotonic decreasing function $f(x_k)$.

4.2. Linear system, linear measurement, and uniform disturbances

We wish to study the following linear system with a linear measurement equation:

$$x_{k+1} = 1.0x_k + w_k, \tag{76}$$

$$z_k = 0.5x_k + v_k. \tag{77}$$

The disturbances w_k and v_k have uniformly shaped membership functions such that $a_w(k) = -3.0$, $b_w(k) = 3.0$, $a_v(k) = -0.5$, and $b_v(k) = 0.5$ for all k. Recall that the variance of a uniform density function $f(x) = (1/(b-a))\{u(x-a) - u(x-b)\}$ is given by $(b-a)^2/12$.

In this study we performed 20 Monte Carlo runs of the above system where a single run consisted of 100 discrete time iterations with measurement updates performed every 10 iterations. Fig. 5(a) shows the results for run number 1 of 20 runs for both the fuzzy estimator and the Kalman filter. Fig. 5(b) shows the 20 run Monte Carlo statistics for both filters. From the results shown in Fig. 5 it is difficult to determine which filter performed the best. In fact the Monte Carlo results show such a small difference that the plots are practically on top of each other. However, Table 1 below shows the energy in the error signal (computed as $\frac{1}{2}e^{\mathrm{T}}e$) for the 20 Monte Carlo runs (boxed numbers in the table indicate cases where the fuzzy estimator outperforms the Kalman filter). Notice that the fuzzy estimator outperforms the Kalman filter 15 out of 20 runs. It is important to point out that this performance is not an isolated case but was typical of the many trial runs of the simulation.

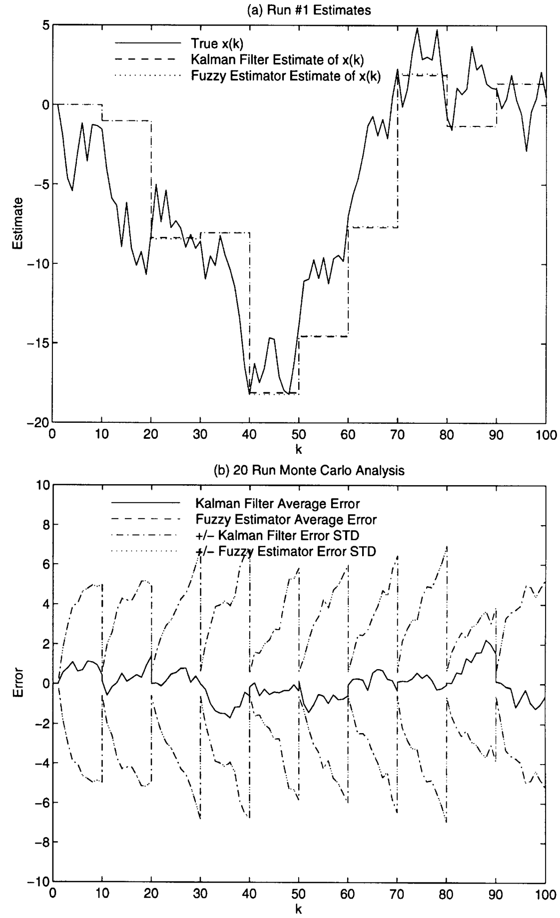

Fig. 5. Linear system, linear measurement, and uniform disturbances: (a) first run results, (b) 20 run Monte Carlo analysis ("STD" is standard deviation).

4.3. Linear system, non-linear measurement, and uniform disturbances

We wish to study the following linear system with a non-linear measurement equation:

$$x_{k+1} = 1.0x_k + w_k, \tag{78}$$

$$z_k = 0.0005x_k^3 + v_k. \tag{79}$$

The disturbances w_k and v_k have uniformly shaped membership functions such that $a_w(k) = -3.0$, $b_w(k) = 3.0$, $a_v(k) = -0.5$, and $b_v(k) = 0.5$ for all k.

In this study we performed 20 Monte Carlo runs of the above system where a single run consisted of 100 discrete time iterations with measurement updates performed every 10 iterations. Fig. 6(a) shows the results for run number 1 of 20 runs for both the fuzzy estimator and the extended Kalman filter. Fig. 6(b) shows the 20 run Monte Carlo statistics for both filters. The Monte Carlo results show that the fuzzy estimator performs significantly better than the extended Kalman filter. Table 2 below shows the energy in the error signal (computed $\frac{1}{2}e^{\mathrm{T}}e$) for the 20 Monte Carlo runs (boxed numbers in the table indicate cases where the fuzzy estimator outperforms the Kalman filter). Notice that the fuzzy estimator outperforms the Kalman filter 15 out of 20 runs. Again this performance is not

Table 1
Energy in error – linear system, linear measurements, and uniform disturbances

Run no.	Kalman filter error energy	Fuzzy estimator error energy
1	0.0206	0.0204
2	0.5384	0.5228
3	0.0035	0.0031
4	0.0148	0.0151
5	0.2295	0.2201
6	0.1003	0.0972
7	0.0202	0.0205
8	0.2908	0.2774
9	1.1933	1.1538
10	0.0024	0.0014
11	0.0058	0.0079
12	1.4112	1.3619
13	2.1468	2.0758
14	0.0280	0.0265
15	0.0104	0.0111
16	0.0440	0.0407
17	0.4370	0.4088
18	0.0138	0.0140
19	0.0412	0.0389
20	0.7661	0.7444

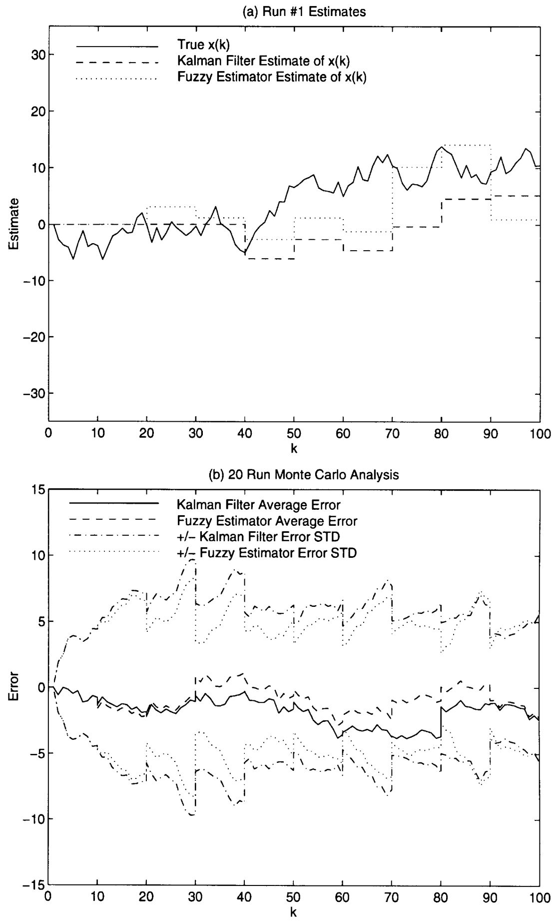

Fig. 6. Linear system, non-linear measurement, and uniform disturbances: (a) first run results, (b) 20 run Monte Carlo analysis ("STD" is standard deviation).

an isolated case but was typical of the many trial runs of the simulation.

4.4. Non-linear system, linear measurement, and uniform disturbances

We wish to study the following non-linear system with a linear measurement equation:

$$x_{k+1} = 0.01x_k^3 + w_k, \tag{80}$$

$$z_k = 1.0x_k + v_k. \tag{81}$$

The disturbances w_k and v_k have uniformly shaped membership functions such that $a_w(k) = -2.0$, $b_w(k) = 2.0$, $a_v(k) = -0.5$, and $b_v(k) = 0.5$ for all k.

In this study we performed 20 Monte Carlo runs of the above system where a single run consisted of 100 discrete time iterations with measurement updates performed every 10 iterations. Fig. 7(a) shows the results for run number 1 of 20 runs for both the fuzzy estimator and the Kalman filter. Fig. 7(b) shows the 20 run Monte Carlo statistics for both filters. From the results shown in Fig. 7 it is difficult to determine which filter performed best. In fact, the Monte Carlo results show such a small difference that the plots are practically on top of each other. However, Table 3 below shows the energy in the error signal (computed as $\frac{1}{2}e^{\mathrm{T}}e$) for the 20 Monte Carlo runs (boxed numbers in the table are used to indicate cases where the fuzzy estimator

Table 2
Energy in error – linear system, non-linear measurements, and uniform disturbances

Run no.	Extended Kalman filter error energy	Fuzzy estimator error energy
1	9.9088	0.9492
2	8.6008	3.2983
3	5.0778	0.7673
4	13.893	1.0984
5	2.0858	0.0001
6	0.6675	0.8350
7	0.3584	1.3149
8	0.0114	4.7174
9	0.1984	0.3036
10	7.0437	0.7604
11	10.549	0.3871
12	2.0992	0.6510
13	8.0853	4.2605
14	14.511	2.8257
15	2.6094	6.5030
16	7.6547	1.7186
17	6.3317	0.5599
18	10.412	4.3276
19	5.8117	0.2005
20	3.9570	0.0394

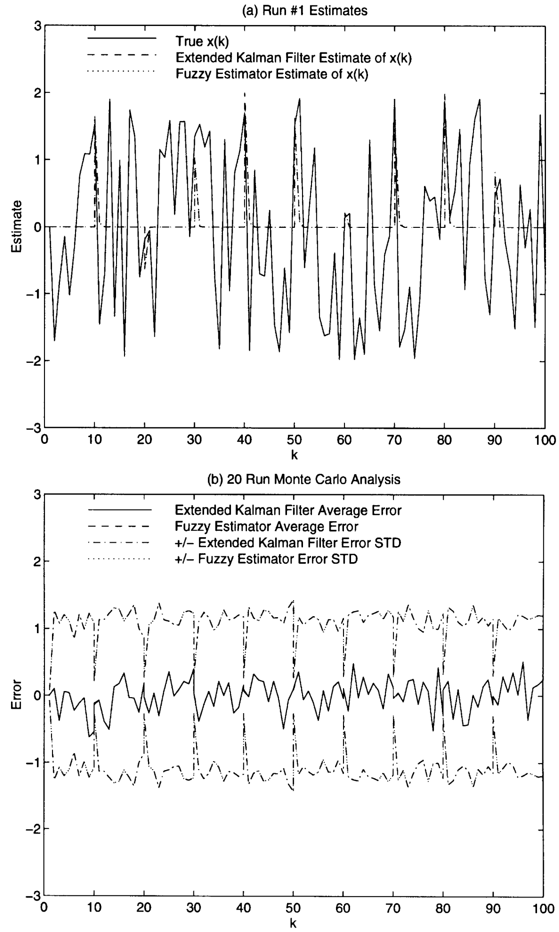

Fig. 7. Non-linear system, linear measurement, and uniform disturbances: (a) first run results, (b) 20 run Monte Carlo analysis ("STD" is standard deviation).

outperforms the extended Kalman filter). Notice that the fuzzy estimator outperforms the Kalman filter 16 out of 20 runs. Again this performance is not an isolated case but was typical of the many trial runs of the simulation.

4.5. Non-linear system, non-linear measurement, and uniform disturbances

We wish to study the following non-linear system with a non-linear measurement equation:

$$x_{k+1} = 0.01x_k^3 + w_k, \tag{82}$$

$$z_k = 1.0x_k^3 + v_k. \tag{83}$$

The disturbances w_k and v_k have uniformly shaped membership functions such that $a_w(k) = -2.0$, $b_w(k) = 2.0$, $a_v(k) = -0.5$, and $b_v(k) = 0.5$ for all k.

In this study we performed 20 Monte Carlo runs of the above system where a single run consisted of 100 discrete time iterations with measurement updates performed every 10 iterations. Fig. 8(a) shows the results for run number 1 of 20 runs for both the fuzzy estimator and the Kalman filter. Fig. 8(b) shows the 20 run Monte Carlo statistics for both filters. From the results shown in Fig. 8 it is difficult to determine which filter performed the best. In fact, the Monte Carlo results show such a small difference that the plots are practically on top of each other. However, Table 4

Table 3
Energy in error – non-linear system, linear measurements, and uniform disturbances ("0.8398e−3" represents the number 0.8398×10^{-3})

Run no.	Extended Kalman filter error energy	Fuzzy estimator error energy
1	0.8398e−3	0.8375e-3
2	0.0098	0.0096
3	0.1869e−3	0.1743e-3
4	0.1955e−3	0.1578e-3
5	0.2385e−4	0.8722e−4
6	0.0089	0.0088
7	0.0085	0.0080
8	0.0146	0.0148
9	0.0044	0.0042
10	0.0203	0.0197
11	0.0057	0.0052
12	0.1605e−4	0.0120e-4
13	0.0015	0.0019
14	0.0067	0.0065
15	0.3056e−3	0.2923e-3
16	0.0030	0.0029
17	0.0018	0.0016
18	0.0024	0.0027
19	0.0018	0.0014
20	0.0052	0.0051

below shows the energy in the error signal (computed as $\frac{1}{2}e^{\mathrm{T}}e$) for the 20 Monte Carlo runs (boxed numbers in the table are used to indicate cases where the fuzzy estimator outperforms the extended Kalman filter). Notice that the fuzzy estimator outperforms the Kalman filter 18 out of 20 runs. Again this performance is not an isolated case but was typical of the many trial runs of the simulation.

5. Rocket launch estimation problem

In previous sections we investigate systems where the propagated membership function maintained a similar shape as the initial membership function for all time. In such cases it is necessary to only keep track of a finite number of parameters that characterize the membership function. In this section we investigate a rocket launch estimation problem where the membership function cannot be characterized by a few parameters because the shape can change significantly over time. Here, it is necessary to discretize the universe of discourse so that we keep track of a finite number of discrete points of the membership function rather than a continuum of points that completely characterizes the membership function.

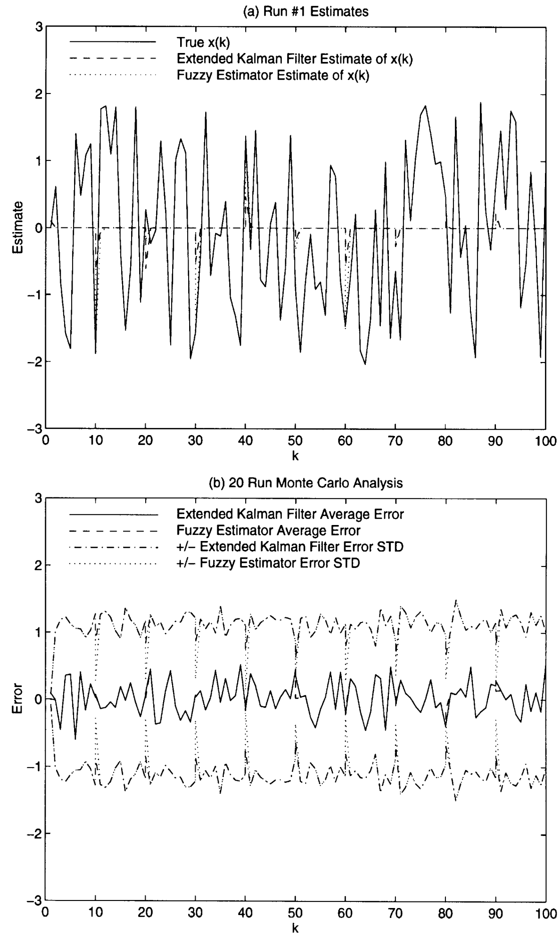

Fig. 8. Non-linear system, non-linear measurement, and uniform disturbances: (a) first run results, (b) 20 run Monte Carlo analysis ("STD" is standard deviation).

Table 4
Energy in error – non-linear system, non-linear measurements, and uniform disturbances ("0.1572e−3" represents the number 0.1572×10^{-3})

Run no.	Extended Kalman filter error energy	Fuzzy estimator error energy
1	0.0013	0.0005
2	0.1572e−3	0.0565e-3
3	0.0219	0.0184
4	0.0223	0.0182
5	0.0237	0.0217
6	0.4345e−3	0.0998e-3
7	0.1471e−4	0.8272e−4
8	0.0059	0.0052
9	0.0027	0.0021
10	0.0099	0.0083
11	0.0052	0.0031
12	0.0018	0.0008
13	0.0021	0.0017
14	0.0054	0.0050
15	0.0022	0.0013
16	0.0012	0.0020
17	0.0058	0.0016
18	0.0028	0.0007
19	0.0116	0.0064
20	0.0050	0.0033

5.1. The fuzzy estimator with discrete universes of discourse: implementation code

Since it may not be clear to the reader how one might implement the fuzzy estimator using discrete universes of discourse, we provide pseudo-code for a second-order stochastic process. Using the code presented here it is easy to generalize these results to higher-order systems.

Consider the following second-order stochastic process:

$$x_1(k+1) = f_1(x_1(k), x_2(k), w(k)), \tag{84}$$

$$x_2(k+1) = f_2(x_1(k), x_2(k)), \tag{85}$$

where $x_1(k)$ and $x_2(k)$ are the system states at time kT and $w(k)$ is a white noise input disturbance. Also, assume that measurements are given by the following equation:

$$z(k) = g(x_1(k), x_2(k), v(k)), \tag{86}$$

where $v(k)$ is a white noise output disturbance.

Fig. 9 shows the main variables that need to be defined for this problem. For example, the first line defines a real array XRELC for the current fuzzy relation state with indices for state x_1 ranging from $-NX1$ to $NX1$ and indices for x_2 ranging from $-NX2$ to $NX2$. When indexing the membership functions defined in Fig. 9, we will use the integer M to index state x_1, N to index state x_2, O to index the input noise w, P to index the output noise v, and Q to index the measurement z. Also, it is important to note that the index corresponds to a specific value on the appropriate universe of discourse. For example, assume that x_1 is defined on the interval $[-10, 10]$. To convert from the index M to a specific value of x_1 we must perform the following multiplication $x_1 = M \cdot M2X1$ where $M2X1 = 10/NX1$. Likewise, similar conversions are made for x_2 with $N2X2$, w with $O2W$, v with $P2V$, and z with $Q2Z$.

Propagation of the fuzzy relation is summarized in the pseudo-code in Fig. 10. This pseudo-code implements the fuzzy dynamic system model of the fuzzy estimator. First we find the membership function for $X \times W$ using the multiply operation for the triangular norm. Then we find the maximum of this membership function subject to the constraints of Eqs. (84) and (85) to determine the next fuzzy state (i.e. we are implementing the fuzzy extension principle). The code in Fig. 10 is repeated for each iteration until a measurement is available to be processed.

Measurement updates of the fuzzy relation for the state in the fuzzy estimator are summarized in the pseudo-code in Fig. 11. This code implements the measurement update equation. First we find the membership function for $X \times V$ using the multiply operation for the triangular norm. Then we find the maximum of this membership function subject to the constraint of Eq. (86) to determine the membership function for $X \times Z$. Next, we use the fuzzy Bayes rule to determine next fuzzy state after measurement update using the

```
REAL XRELC(-NX1:NX1,-NX2:NX2)          CURRENT FUZZY RELATION STATE MEMBERSHIP FUNCTION
REAL XRELN(-NX1:NX1,-NX2:NX2)          NEXT FUZZY RELATION STATE MEMBERSHIP FUNCTION
REAL W(-NW,NW)                         MEMBERSHIP FUNCTION FOR THE INPUT NOISE
REAL V(-NV,NV)                         MEMBERSHIP FUNCTION FOR THE MEASUREMENT NOISE
REAL XXW(-NX1:NX1,-NX2:NX2,-NW:NW)     MEMBERSHIP FUNCTION FOR X × W
REAL XXV(-NX1:NX1,-NX2:NX2,-NV:NV)     MEMBERSHIP FUNCTION FOR X × V
REAL XXZ(-NX1:NX1,-NX2:NX2,-NZ:NZ)     MEMBERSHIP FUNCTION FOR X × Z
REAL XF1, XF2                          FUZZY ESTIMATE OF X1 AND X2, RESPECTIVELY
REAL Z                                 THE SYSTEM MEASUREMENT
```

Fig. 9. Variables that need to be defined to implement the fuzzy estimator ("REAL" indicates that the variable is a real number).

```
FOR M = -NX1:NX1                                              FIND X × W
     FOR N = -NX2:NX2
          FOR O = -NW:NW
               XXW(M,N,O) = XRELC(M,N)*W(O)
          END FOR
     END FOR
END FOR
FOR M = -NX1:NX1                                              FIND XRELN
     FOR N = -NX2:NX2
          FOR O = -NW:NW
               MM = ROUND[(f1(M*M2X1,N*N2X2,O*O2W)/M2X1)]    FIND THE INDICES FOR XRELN
               NN = ROUND[(f2(M*M2X1,N*N2X2)/N2X1)]
               IF (XXW(M,N,O) ≥ XXW(MM,NN)) THEN              CHECK TO SEE IF IT IS A MAXIMUM
                    XRELN(MM,NN) = XXW(M,N,O)
               END IF
          END FOR
     END FOR
END FOR
```

Fig. 10. Pseudo-code for propagation of the fuzzy relation state in the fuzzy estimator.

current measurement. Then we normalize the result so that its maximum value occurs at one.

To use the fuzzy estimator's result (i.e. XRELN) in applications it is sometimes necessary to defuzzify the fuzzy relation to some crisp value. For the results presented in this dissertation we used the center of gravity defuzzification technique summarized in pseudo-code in Fig. 12.

5.2. Computational complexity

Notice that for the fuzzy relation propagation, the measurement update equations, and the center of gravity defuzzification routines we must completely loop through the fuzzy relation array for the state one or more times. This leads to significant computational complexity. It is easy to see that the fuzzy relation propagation for the fuzzy state requires $16 * NX1 * NX2 * NW$ iterations. Likewise, the number of iterations for measurement updates is $8 * NX1 * NX2 * (2 * NV + 1)$ and the number of iterations for a center of gravity defuzzification is $4 * NX1 * NX2$. Notice that propagation and measurement update require on the order of $16 * NX1 * NX2 * NW$ computations while defuzzification is a relatively small number by comparison.

Assume $NX1 = NX2 = NW = NV = 1000$ then both propagation and measurement update require approximate $16(1000)^3 = 1.6 \times 10^{10}$ iterations which is quite large. If we increase to third order system this becomes $16(1000)^4 = 1.6 \times 10^{13}$. Notice that it quickly becomes too large to solve reasonably even on a modern digital computer. Clearly, the computational complexity is a serious limitation when implementing the fuzzy estimator. However, it may be possible to combine several operations to optimize the code. Furthermore, the computational complexity may become less of a constraint as computers are quickly evolving to higher speeds and larger memories.

Next we develop a fuzzy estimator for a rocket launch problem.

5.3. Rocket model

A mathematical model for a single stage rocket is presented by Barrére et al. [23] and Mandell et al.

```
FOR M = -NX1:NX1                                          FIND X × V
     FOR N = -NX2:NX2
          FOR P = -NV:NV
               XXV(M,N,P) = XRELC(M,N)*V(P)
          END FOR
     END FOR
END FOR
FOR M = -NX1:NX1                                          FIND X × Z
     FOR N = -NX2:NX2
          FOR P = -NV:NV
               Q = ROUND[(g(M*M2X1,N*N2X2,P*P2V)/Q2Z)]
               IF (XXV(M,N,O) ≥ XXZ(M,N,Q)) THEN
                    XXZ(M,N,Q) = XXV(M,N,O)
               END IF
          END FOR
     END FOR
END FOR
Q = ROUND(Z/Q2Z)                                          CONVERT THE MEASUREMENT TO AN INDEX
BIGGEST = 0.0                                             BIGGEST IS USED FOR NORMALIZATION
FOR M = -NX1:NX1                                          FIND THE NON-NORMALIZED XRELN
     FOR N = -NX2:NX2
          XRELN(M,N) = XXZ(M,N,Q)
          IF (XRELN(M,N) ≥ BIGGEST) THEN
               BIGGEST = XRELN(M,N)
          END IF
     END FOR
END FOR
FOR M = -NX1:NX1                                          NORMALIZED XRELN
     FOR N = -NX2:NX2
          XRELN(M,N) = XRELN(M,N)/BIGGEST
     END FOR
END FOR
```

Fig. 11. Pseudo-code for measurement updates of the fuzzy relation state in the fuzzy estimator.

[24] and is expressed by the following differential equation:

$$\ddot{h}(t) = c(t)\left(\frac{m}{M - mt}\right) - g_o\left(\frac{R}{R + h(t)}\right) - 0.5\,\dot{h}^2(t)\left(\frac{\rho_a A C_d}{M - mt}\right) + w(t), \quad (87)$$

where $h(t)$ is the altitude of the rocket (above sea level) at time t, $w(t)$ is a disturbance resulting from, for example, gravitational anomalies, and for our simulations the rocket parameters are:

1. $M = 20100.00$ kg — initial mass of the rocket and fuel,
2. $m = 100.0$ kg/s — exhaust gases mass flow rate (approximately constant for some solid propellant rockets),
3. $A = 1.0$ m^2 — maximum cross sectional area of the rocket,
4. $g_o = 9.8$ m/s^2 — the acceleration due to gravity at sea level,
5. $R = 6.37 \times 10^6$ m — radius of the earth,

```
SUM = 0.0
SUMM = 0.0
SUMN = 0.0
FOR M = -NX1:NX1
     FOR N = -NX2:NX2
          SUM = SUM + XRELC(M,N)
          SUMM = SUMM + XRELC(M,N)*M*M2X1
          SUMN = SUMN + XRELC(M,N)*N*N2X2
     END FOR
END FOR
XF1 = SUMM/SUM
XF2 = SUMN/SUM
```

Fig. 12. Pseudo-code for the center of gravity defuzzification.

6. $\rho_a = 1.21$ kg/m^3 — density of air,
7. $C_d = 0.3$ — drag coefficient for the rocket,
8. $c(t) = 4000.0$ m/s — the velocity of the exhaust gases.

The rocket is assumed to have an altimeter which provides measurements of the rocket altitude (e.g. a baro altimeter). Here, we model the altitude measurement, denoted $z(t)$, as the sum of the true altitude, $h(t)$, and a measurement disturbance, denoted $v(t)$, i.e.

$$z(t) = h(t) + v(t). \quad (88)$$

For our simulation measurements are assumed to be taken every 20 s.

The mathematical model in Eq. (87) was developed based on the simple dynamics of a point mass. However, in general, rockets dynamics are studied in the realm of exterior ballistics. This type of analysis often tends to be very complex and falls outside the scope of this article. However, even in this restricted context the modeled dynamics provide a complex example for state estimation.

For example, due to the loss of fuel resulting from the combustion and exhaust the rocket has a time-varying mass. Furthermore, it can be determined by inspection of Eq. (87) that the system is a non-linear process. The primary purpose for considering this application is to investigate the capability of the fuzzy dynamic estimator for estimating states of highly non-linear, time-varying systems. Furthermore, we will compare these results to that of the extended Kalman filter.

5.4. Discretization of the Rocket Model

Our analysis in this article has been constrained to discrete-time systems. Therefore, the model presented in the previous subsection must be discretized so that the fuzzy dynamic model based estimator can be implemented. Here, we use a simple Euler integration technique to discretize the system. Hence, the discretized system dynamics are given by

$$x_1(k+1) = x_1(k) + Tx_2(k), \tag{89}$$

$$x_2(k+1) = x_2(k) + T\left[\left(\frac{c(k)m}{M - mTk}\right) - \left(\frac{g_o R}{R + x_1(k)}\right) - \left(\frac{0.5x_2^2(k)\rho_a A C_d}{M - mTk}\right)\right] + w_d(k), \tag{90}$$

where T is the sample period, $x_1(k)$ is the rocket altitude at time kT, $x_2(k)$ is the rocket velocity at time kT, and $w_d(k)$ is the discretized noise at time kT. We found that a sample period of $T = 2$ s produced accurate results. The discretized measurement equation does not change. Measurements are given by

$$z(k) = x_1(k) + v(k). \tag{91}$$

For the simulation, we assume that $w_d(t)$ and $v(t)$ are white noise disturbances with Cauchy probability density functions; hence we choose the *membership functions* that characterize these as

$$\mu_W(w_d) = \frac{1}{1 + (w_d/15)^2}, \tag{92}$$

$$\mu_V(v) = \frac{1}{1 + (w_d/1000)^2}. \tag{93}$$

5.5. Extended Kalman filter design

In designing the extended Kalman filter, Eqs. (89)–(91) must be linearized to obtain

$$\boldsymbol{F}(\tilde{\boldsymbol{x}}_k, k) = \left.\frac{\partial \boldsymbol{f}(\boldsymbol{x}_k, k)}{\partial \boldsymbol{x}_k}\right|_{\boldsymbol{x}_k = \tilde{\boldsymbol{x}}_k} = \begin{bmatrix} 1 & T \\ \dfrac{Tg_o R}{(R + \tilde{x}_1(k))^2} & 1 - \dfrac{T\tilde{x}_2(k)\rho_a A C_d}{M - mkT} \end{bmatrix}, \tag{94}$$

$$\boldsymbol{G}(\tilde{\boldsymbol{x}}_k, k) = \left.\frac{\partial \boldsymbol{g}(\boldsymbol{x}_k, k)}{\partial \boldsymbol{x}_k}\right|_{\boldsymbol{x}_k = \tilde{\boldsymbol{x}}_k} = [1 \quad 0]. \tag{95}$$

Also, we need to define the covariance matrices Q and R. This choice was difficult since the covariance of a Cauchy distribution is undefined. We tried various choices for the variance. The following choice for the Q and R matrices produced good results:

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 15^2 \end{bmatrix}, \tag{96}$$

$$R = 1000^2. \tag{97}$$

5.6. Fuzzy estimator design

In designing the fuzzy estimator, we discretized the universes of discourse so that each state's membership function is represented by 1000 discrete points. Since there are two states in this problem, this means that the fuzzy relation characterizing these states are represented by a 1000×1000 array. The 1000 points

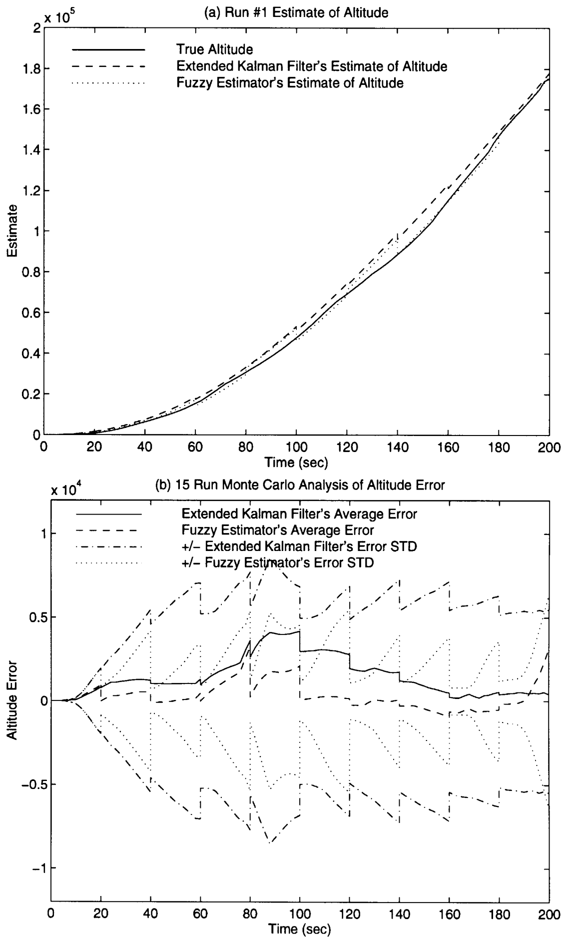

Fig. 13. Rocket altitude estimates with Cauchy disturbances: (a) first run results, (b) 15 run Monte Carlo analysis ("STD" is standard deviation).

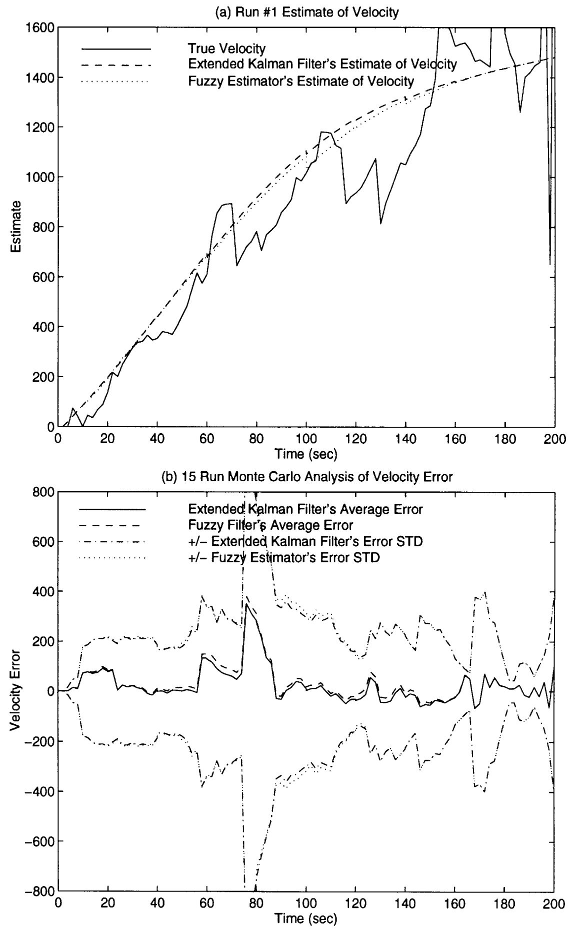

Fig. 14. Rocket velocity estimates with Cauchy disturbances: (a) first run results, (b) 15 run Monte Carlo analysis ("STD" is standard deviation).

for the rocket altitude discretizes altitudes on the interval $[0, 200\,000]$. Likewise, the 1000 points for altitude velocity discretizes the set of velocities on the interval $[0, 2000]$.

The membership functions for the noise process noise, $w_d(t)$, is discretized by 120 points which represent noises values on the interval $[-15, 15]$. Likewise, the membership function for the measurement noise, $v(t)$, is discretized by 120 points which represent values on the interval $[-1000, 1000]$.

The center of gravity defuzzification technique was used to convert the fuzzy relation state to a "crisp" state estimate.

5.7. Simulation results

We simulated the rocket for 200 s. Measurements were provided to the Kalman filter and the fuzzy estimator every 20 s. Hence, we had 10 measurements for every run. We simulated 15 such launches of the rocket.

The initial conditions for the extended Kalman filter for each run were zero for both the altitude and the velocity. Since we know that the rocket is launched from a stationary position at ground level, the extended Kalman filter's covariance matrix was set to zero. The fuzzy estimator's state was set to a fuzzy singleton at zero altitude and zero velocity.

Table 5
Energy in altitude error – rocket launch with Cauchy disturbances ("1.1069e9" represents the number 1.1069×10^9)

Run no.	Kalman filter error energy	Fuzzy estimator error energy
1	1.1069e9	0.2710e9
2	3.2948e9	0.5430e9
3	0.7444e9	0.7550e9
4	0.8057e9	0.3737e9
5	0.6832e9	0.2153e9
6	0.9067e9	0.2388e9
7	1.0646e9	0.2078e9
8	2.5950e9	1.4402e9
9	3.5383e9	1.0915e9
10	4.8707e9	0.4013e9
11	0.6368e9	0.1299e9
12	1.9402e9	0.4322e9
13	0.1577e9	0.0267e9
14	0.8353e9	0.2032e9
15	4.1994e9	0.7887e9

Table 6
Energy in velocity error – rocket launch with Cauchy disturbances ("1.1069e9" represents the number 1.1069×10^9)

Run no.	Kalman filter error energy	Fuzzy estimator error energy
1	0.3771e7	0.3622e7
2	0.4791e7	0.4683e7
3	0.5573e7	0.6107e7
4	0.4758e7	0.4737e7
5	0.1169e7	0.1167e7
6	0.1536e7	0.1698e7
7	0.1538e7	0.1526e7
8	3.3999e7	3.4159
9	0.1945e7	0.1416e7
10	0.4445e7	0.4262e7
11	0.1680e7	0.1815e7
12	0.2024e7	0.1959e7
13	0.0514e7	0.0542e7
14	0.1828e7	0.1815e7
15	0.4466e7	0.4410e7

Figs. 13(a) and 14(a) show the altitude and velocity results, respectively, for run number 1 of 15 runs for both the fuzzy estimator and the extended Kalman filter. Figs. 13(b) and 14(b) show the 15 run Monte Carlo statistics for both filters. From the Monte Carlo results for altitude error it is clear that the fuzzy filter outperformed the extended Kalman filter. The Monte Carlo result for the velocity error is not as conclusive.

Tables 5 and 6 show the energy in the altitude and velocity error, respectively, (computed as $\frac{1}{2}e^{\mathrm{T}}e$) for the 15 Monte Carlo runs (boxed numbers are used in the tables to indicate cases where the fuzzy estimator outperforms the Kalman filter). Notice that the fuzzy estimator outperforms the Kalman filter 14 out of 15 runs for the altitude and 10 out of 15 runs for the velocity. These results show that the fuzzy estimator is outperforming the Kalman filter in both the altitude and velocity estimates.

6. Concluding remarks

In this article we have presented a state estimator that was developed using fuzzy dynamic system models. We have shown that for linear systems and Gaussian-shaped membership functions the fuzzy state estimator produces the exact same result as the Kalman filter. Consequently, we believe it is possible to find other classes of systems whose solution is difficult to obtain with probabilistic methods but easily obtained using the fuzzy estimator approach. For example, we investigated and compared simulation results for several first order stochastic systems driven by uniform distributed white noise. These results included a combination of both linear and non-linear propagation and measurement equations. For each system studied we found that the fuzzy estimator outperformed the extended Kalman filter. Finally, we showed that the fuzzy estimator outperformed the extended Kalman filter for a rocket launch system driven by white Cauchy distributed noise.

In [15] it is shown that convergence of the Kalman filter's estimate to the true value is guaranteed only when the system is stochastically controllable and observable. Therefore, one future research direction is to define and characterize these concept for fuzzy dynamic system models. Clearly another area for future study is the development of efficient computational strategies for the fuzzy estimator.

Acknowledgements

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Appendix A . Proofs of theorems

A.1. Proof of state propagation equation, Theorem 3.1

The problem of finding the state propagation equation is to solve the following optimization problem obtained by application of the fuzzy extension principle.

Maximize

$$\mu_{\bar{X}(k)\times W(k)}(x_k, w_k) = \exp\{-\tfrac{1}{2}[(x_k - \tilde{x}_k)^{\mathrm{T}} P_k^{-1}(x_k - \tilde{x}_k)] - \tfrac{1}{2}(w_k)^{\mathrm{T}} Q^{-1}(w_k)]\} \tag{A.1}$$

subject to

$$x_{k+1} = Ax_k + w_k. \tag{A.2}$$

We will let P_k and Q be positive definite symmetric matrices. Hence, we have the following relationships

$$P_k = P_k^{\mathrm{T}}, \tag{A.3}$$

$$P_k^{-\mathrm{T}} = P_k^{-1}, \tag{A.4}$$

$$Q = Q^{\mathrm{T}}, \tag{A.5}$$

$$Q^{-\mathrm{T}} = Q^{-1}. \tag{A.6}$$

Begin by forming the Lagrangian

$$L = \exp\{-\tfrac{1}{2}[(x_k - \tilde{x}_k)^{\mathrm{T}} P_k^{-1}(x_k - \tilde{x}_k)] - \tfrac{1}{2}[(w_k)^{\mathrm{T}} Q^{-1}(w_k)]\} - \lambda^{\mathrm{T}}[Ax_k + w_k - x_{k+1}]. \tag{A.7}$$

Recall the following:

$$\frac{\partial(\underline{x}^{\mathrm{T}} A \underline{x})}{\partial \underline{x}} = 2A\underline{x}, \tag{A.8}$$

$$\frac{\partial(\underline{x}^{\mathrm{T}} A \underline{y})}{\partial \underline{x}} = A\underline{y}, \tag{A.9}$$

$$\frac{\partial(\underline{x}^{\mathrm{T}} A \underline{y})}{\partial \underline{y}} = A^{\mathrm{T}}\underline{x}. \tag{A.10}$$

Take the partial derivatives of the Lagrangian in Eq. (A.7)

$$\frac{\partial L}{\partial x_k} = -P_k^{-1}(x_k - \tilde{x}_k)\mu_{X(k)\times W(k)}(x_k, w_k) - A^{\mathrm{T}}\lambda = 0, \tag{A.11}$$

$$\frac{\partial L}{\partial w_k} = -Q^{-1} w_k \mu_{X(k)\times W(k)}(x_k, w_k) - \lambda = 0, \tag{A.12}$$

$$\frac{\partial L}{\partial \lambda} = -[Ax_k + w_k - x_{k+1}] = 0. \tag{A.13}$$

From Eqs. (A.11) and (A.12) we obtain

$$\lambda = -A^{-\mathrm{T}} P_k^{-1}(x_k - \tilde{x}_k)\mu_{X(k)\times W(k)}(x_k, w_k) \tag{A.14}$$

$$= -Q^{-1} w_k \mu_{X(k)\times W(k)}(x_k, w_k). \tag{A.15}$$

Notice that this equation requires that A^{T} be invertible. Rearranging this equation yields

$$(x_k - \tilde{x}_k) = P_k A^{\mathrm{T}} Q^{-1} w_k. \tag{A.16}$$

From Eqs. (A.13) and (A.16) we get

$$(x_k - \tilde{x}_k) = P_k A^{\mathrm{T}} Q^{-1}[x_{k+1} - Ax_k]. \tag{A.17}$$

Rearranging this equation yields

$$x_k = [I + P_k A^{\mathrm{T}} Q^{-1} A]^{-1}[P_k A^{\mathrm{T}} Q^{-1} x_{k+1} + \tilde{x}_k]. \tag{A.18}$$

If Q^{-1} is positive definite (which we assume) and A is non-singular (which we assume), the matrix $A^{\mathrm{T}}Q^{-1}A$

is symmetric positive definite [8]. Further, since product of two symmetric matrices is symmetric, then $P_kA^{\mathrm{T}}Q^{-1}A$ is symmetric and due to [8] also positive definite. Hence, the inverse $[I+P_kA^{\mathrm{T}}Q^{-1}A]^{-1}$ in Eq. (A.18) exists. Notice that

$$\begin{aligned} x_k - \tilde{x}_k &= [I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1} \\ &\quad \times [P_kA^{\mathrm{T}}Q^{-1}x_{k+1} + \tilde{x}_k] - \tilde{x}_k, && (A.19) \\ &= [I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1}[P_kA^{\mathrm{T}}Q^{-1}x_{k+1} + \tilde{x}_k \\ &\quad - [I + P_kA^{\mathrm{T}}Q^{-1}A]\tilde{x}_k], && (A.20) \\ &= [I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1}(P_kA^{\mathrm{T}}Q^{-1}) \\ &\quad \times [x_{k+1} - A\tilde{x}_k]. && (A.21) \end{aligned}$$

From Eqs. (A.13) and (A.18) we get

$$\begin{aligned} w_k &= x_{k+1} - Ax_k, && (A.22) \\ &= x_{k+1} - A[I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1} \\ &\quad \times [P_kA^{\mathrm{T}}Q^{-1}x_{k+1} + \tilde{x}_k] && (A.23) \\ &= A[I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1}\{[I + P_kA^{\mathrm{T}}Q^{-1}A] \\ &\quad \times A^{-1}x_{k+1} - P_kA^{\mathrm{T}}Q^{-1}x_{k+1} - \tilde{x}_k\} && (A.24) \\ &= A[I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1}A^{-1} \\ &\quad \times [x_{k+1} - A\tilde{x}_k]. && (A.25) \end{aligned}$$

Again notice that we need A^{-1} to exist. Substitute Eqs. (A.21) and (A.25) into Eq. (A.1) to get $\mu_{\tilde{X}(k+1)}(x_{k+1})$ from the fuzzy extension principle

$$\mu_{\tilde{X}(k+1)}(x_{k+1}) = \exp\{-\tfrac{1}{2}(x_{k+1} - A\tilde{x}_k)^{\mathrm{T}} \times P_{k+1}^{-1}(x_{k+1} - A\tilde{x}_k)\}, \quad (A.26)$$

where

$$\begin{aligned} P_{k+1}^{-1} &= \{[I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1}[P_kA^{\mathrm{T}}Q^{-1}]\}^{\mathrm{T}} \\ &\quad \times P_k^{-1}\{[I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1}[P_kA^{\mathrm{T}}Q^{-1}]\} \\ &\quad + \{A[I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1}A^{-1}\}^{\mathrm{T}} \\ &\quad \times Q^{-1}\{A[I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1}A^{-1}\}. && (A.27) \end{aligned}$$

The remaining portion of this proof focuses on simplifying this equation.

Recall the matrix inversion lemma (MIL), where if all the indicated inverses exist,

$$\begin{aligned} &[\bar{A} + \bar{B}\bar{D}\bar{C}]^{-1} \\ &\quad = \bar{A}^{-1} - \bar{A}^{-1}\bar{B}[\bar{D}^{-1} + \bar{C}\bar{A}^{-1}\bar{B}]^{-1}\bar{C}\bar{A}^{-1}. && (A.28) \end{aligned}$$

Using the matrix inversion lemma with $\bar{A}=I$, $\bar{B}=A^{\mathrm{T}}Q^{-1}A$, $\bar{C}=P_k$, and $\bar{D}=I$ we get

$$\begin{aligned} &[I + P_kA^{\mathrm{T}}Q^{-1}A]^{-\mathrm{T}} \\ &\quad = [I + A^{\mathrm{T}}Q^{-1}AP_k]^{-1} && (A.29) \\ &\quad = I - A^{\mathrm{T}}Q^{-1}A(I + P_kA^{\mathrm{T}}Q^{-1}A)^{-1}P_k. && (A.30) \end{aligned}$$

Substitute Eq. (A.30) into Eq. (A.27) to obtain

$$\begin{aligned} P_{k+1}^{-1} &= [Q^{-1}AP_k][I - A^{\mathrm{T}}Q^{-1}A(I + P_kA^{\mathrm{T}}Q^{-1}A)^{-1}P_k] \\ &\quad \times P_k^{-1}[I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1}[P_kA^{\mathrm{T}}Q^{-1}] \\ &\quad + A^{-\mathrm{T}}[I - A^{\mathrm{T}}Q^{-1}A(I + P_kA^{\mathrm{T}}Q^{-1}A)^{-1}P_k] \\ &\quad \times A^{\mathrm{T}}Q^{-1}A[I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1}A^{-1}. && (A.31) \end{aligned}$$

Rearranging the above equation yields

$$\begin{aligned} P_{k+1}^{-1} &= [Q^{-1}A][I - P_kA^{\mathrm{T}}Q^{-1}A(I + P_kA^{\mathrm{T}}Q^{-1}A)^{-1}] \\ &\quad \times [I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1}[P_k][A^{\mathrm{T}}Q^{-1}] \\ &\quad + A^{-\mathrm{T}}[I - AQ^{-1}A(I + P_kA^{\mathrm{T}}Q^{-1}A)^{-1}P_k] \\ &\quad \times [A^{\mathrm{T}}Q^{-1}A][I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1}A^{-1}. && (A.32) \end{aligned}$$

$$\begin{aligned} P_{k+1}^{-1} &= [Q^{-1}A][I - P_kA^{\mathrm{T}}Q^{-1}A(I + P_kA^{\mathrm{T}}Q^{-1}A)^{-1}] \\ &\quad \times [I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1}[P_k][A^{\mathrm{T}}Q^{-1}] \\ &\quad + [Q^{-1}A][I - (I + P_kA^{\mathrm{T}}Q^{-1}A)^{-1}P_kA^{\mathrm{T}}Q^{-1}A] \\ &\quad \times [I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1}A^{-1}. && (A.33) \end{aligned}$$

From applying the matrix inversion lemma in Eq. (A.28) we find

$$\begin{aligned} &[I + P_kA^{\mathrm{T}}Q^{-1}A]^{-1} \\ &\quad = [I - P_kA^{\mathrm{T}}Q^{-1}A(I + P_kA^{\mathrm{T}}Q^{-1}A)^{-1}] && (A.34) \\ &\quad = [I - (I + P_kA^{\mathrm{T}}Q^{-1}A)^{-1}P_kA^{\mathrm{T}}Q^{-1}A], && (A.35) \end{aligned}$$

where $\bar{A}=I$, $\bar{B}=P_kA^{\mathrm{T}}Q^{-1}A$, $\bar{C}=I$, and $\bar{D}=I$ in Eq. (A.34) and $\bar{A}=I$, $\bar{B}=I$, $\bar{C}=P_kA^{\mathrm{T}}Q^{-1}A$, and $\bar{D}=I$ in Eq. (A.35). Substituting Eqs. (A.34) and (A.35) into Eq. (A.33) yields

$$P_{k+1}^{-1}=[Q^{-1}A][I+P_kA^{\mathrm{T}}Q^{-1}A]^{-2}[P_kA^{\mathrm{T}}Q^{-1}] + [Q^{-1}A][I+P_kA^{\mathrm{T}}Q^{-1}A]^{-2}A^{-1} \quad (A.36)$$

$$=[Q^{-1}A][I+P_kA^{\mathrm{T}}Q^{-1}A]^{-2} \times [P_kA^{\mathrm{T}}Q^{-1}+A^{-1}] \quad (A.37)$$

$$=[Q^{-1}A][I+P_kA^{\mathrm{T}}Q^{-1}A]^{-2} \times [I+P_kA^{\mathrm{T}}Q^{-1}A]A^{-1} \quad (A.38)$$

$$=[Q^{-1}A][I+P_kA^{\mathrm{T}}Q^{-1}A]^{-1}A^{-1} \quad (A.39)$$

$$=[Q^{-1}A][I+P_kA^{\mathrm{T}}Q^{-1}A]^{-1}[(I+P_kA^{\mathrm{T}}Q^{-1}A) - P_kA^{\mathrm{T}}Q^{-1}A]A^{-1} \quad (A.40)$$

$$=[Q^{-1}A][I-[I+P_kA^{\mathrm{T}}Q^{-1}A]^{-1} \times P_kA^{\mathrm{T}}Q^{-1}A]A^{-1} \quad (A.41)$$

$$=Q^{-1}-Q^{-1}A[I+P_kA^{\mathrm{T}}Q^{-1}A]^{-1}P_kA^{\mathrm{T}}Q^{-1} \quad (A.42)$$

$$=[Q+AP_kA^{\mathrm{T}}]^{-1}, \quad (A.43)$$

where the last step was obtained via the matrix inversion lemma (with $\bar{A}=Q$, $\bar{B}=A$, $\bar{C}=P_kA^{\mathrm{T}}$, and $\bar{D}=I$). Substituting Eq. (A.43) into Eq. (A.26) yields

$$\mu_{\tilde{X}(k+1)}(x_{k+1})=\exp\{-\tfrac{1}{2}(x_{k+1}-\tilde{x}_{k+1})^{\mathrm{T}} \times P_{k+1}^{-1}(x_{k+1}-\tilde{x}_{k+1})\}, \quad (A.44)$$

where

$$\tilde{x}_{k+1}=A\tilde{x}_k, \quad (A.45)$$

$$P_{k+1}=AP_kA^{\mathrm{T}}+Q. \quad (A.46)$$

Eqs. (A.45) and (A.46) are recognized as the Kalman filter propagation equations [15].

A.2. Proof of measurement update equation, Theorem 3.2

Since fuzzy composition requires employing a max operation, the problem of finding the state propagation equation is to solve the following optimization problem:

Maximize

$$\mu_{\tilde{X}(k)\times V(k)}(x_k,v_k)=\exp\{-\tfrac{1}{2}[(x_k-\tilde{x}_k)^{\mathrm{T}}P_k^{-1}(x_k-\tilde{x}_k)] -\tfrac{1}{2}[(v_k)^{\mathrm{T}}R^{-1}(v_k)]\} \quad (A.47)$$

subject to

$$z_k=Hx_k+v_k. \quad (A.48)$$

Notice that if we are given a specific z_k and x_k then v_k is uniquely specified by

$$v_k=z_k-Hx_k. \quad (A.49)$$

At first glance this problem appears similar to the one obtained for the propagation equations. However, in this problem we are maximizing Eq. (A.47) with respect to v_k only and not both z_k and v_k. This is done simply by substituting Eq. (A.49) into Eq. (A.47) to obtain

$$\mu_{\tilde{X}(k)\times Z(k)}(x_k,z_k)$$

$$=\exp\{-\tfrac{1}{2}[(x_k-\tilde{x}_k)^{\mathrm{T}}P_k^{-1}(x_k-\tilde{x}_k)]\} \times \exp\{-\tfrac{1}{2}[(z_k-Hx_k)^{\mathrm{T}}R^{-1}(z_k-Hx_k)]\}$$

$$=\exp\{-\tfrac{1}{2}[x_k^{\mathrm{T}}P_k^{-1}x_k-x_k^{\mathrm{T}}P_k^{-1}\tilde{x}_k -\tilde{x}_k^{\mathrm{T}}P_k^{-1}x_k+\tilde{x}_k^{\mathrm{T}}P_k^{-1}\tilde{x}_k]\} \times \exp\{-\tfrac{1}{2}[(z_k^{\mathrm{T}}R^{-1}z_k-z_k^{\mathrm{T}}R^{-1}Hx_k -x_k^{\mathrm{T}}H^{\mathrm{T}}R^{-1}z_k-x_k^{\mathrm{T}}H^{\mathrm{T}}R^{-1}Hx_k)]\}$$

$$=\exp\{-\tfrac{1}{2}[x_k^{\mathrm{T}}(P_k^{-1}+H^{\mathrm{T}}R^{-1}H)x_k -x_k^{\mathrm{T}}(P_k^{-1}\tilde{x}_k+H^{\mathrm{T}}R^{-1}z_k)]\} \times \exp\{-\tfrac{1}{2}[-(\tilde{x}_k^{\mathrm{T}}P_k^{-1}+z_k^{\mathrm{T}}R^{-1}H)x_k +\tilde{x}_k^{\mathrm{T}}P_k^{-1}\tilde{x}_k+z_k^{\mathrm{T}}R^{-1}z_k]\}. \quad (A.50)$$

Define

$$\Gamma = x_k^{\mathrm{T}}(P_k^{-1} + H^{\mathrm{T}}R^{-1}H)x_k - x_k^{\mathrm{T}}(P_k^{-1}\tilde{x}_k + H^{\mathrm{T}}R^{-1}z_k) - (\tilde{x}_k^{\mathrm{T}}P_k^{-1} + z_k^{\mathrm{T}}R^{-1}H)x_k, \tag{A.51}$$

and substitute into Eq. (A.50) to obtain

$$\begin{aligned}\mu_{\tilde{X}(k)\times Z(k)}(x_k, z_k) &= \exp\{-\tfrac{1}{2}[\Gamma + \tilde{x}_k^{\mathrm{T}}P_k^{-1}\tilde{x}_k + z_k^{\mathrm{T}}R^{-1}z_k]\}\\ &= \exp\{-\tfrac{1}{2}[\Gamma + \tilde{x}_k^{\mathrm{T}}P_k^{-1}\tilde{x}_k + z_k^{\mathrm{T}}R^{-1}z_k]\}\\ &\quad\times \exp\{-\tfrac{1}{2}[-(z_k - H\tilde{x}_k)^{\mathrm{T}}(R + HP_kH^{\mathrm{T}})^{-1}\\ &\quad\times (z_k - H\tilde{x}_k)]\}\\ &\quad\times \exp\{-\tfrac{1}{2}[(z_k - H\tilde{x}_k)^{\mathrm{T}}(R + HP_kH^{\mathrm{T}})^{-1}\\ &\quad\times (z_k - H\tilde{x}_k)]\}.\end{aligned} \tag{A.52}$$

The matrix $[R + HP_kH^{\mathrm{T}}]$ is invertible because R and P_k are positive-definite matrices. Define the following:

$$\begin{aligned}\Lambda &= \tilde{x}_k^{\mathrm{T}}P_k^{-1}\tilde{x}_k + z_k^{\mathrm{T}}R^{-1}z_k - (z_k - H\tilde{x}_k)^{\mathrm{T}}\\ &\quad\times (R + HP_kH^{\mathrm{T}})^{-1}(z_k - H\tilde{x}_k) \qquad\qquad (\mathrm{A.53})\\ &= \tilde{x}_x^{\mathrm{T}}[P_k^{-1} - H^{\mathrm{T}}(R + HP_kH^{\mathrm{T}})^{-1}H]\tilde{x}_k\\ &\quad + z_k^{\mathrm{T}}[R^{-1} - (R + HP_kH^{\mathrm{T}})^{-1}]z_k\\ &\quad + z_k^{\mathrm{T}}(R + HP_kH^{\mathrm{T}})^{-1}H\tilde{x}_k\\ &\quad + \tilde{x}_k^{\mathrm{T}}H^{\mathrm{T}}(R + HP_kH^{\mathrm{T}})^{-1}z_k. \qquad\qquad (\mathrm{A.54})\end{aligned}$$

Then Eq. (A.52) becomes

$$\begin{aligned}\mu_{\tilde{X}(k)\times Z(k)}(x_k, z_k) &= \exp\{-\tfrac{1}{2}[\Gamma + \Lambda]\}\\ &\quad\times \exp\{-\tfrac{1}{2}[(z_k - H\tilde{x}_k)^{\mathrm{T}}(R + HP_kH^{\mathrm{T}})^{-1}\\ &\quad\times (z_k - H\tilde{x}_k)]\}.\end{aligned} \tag{A.55}$$

From the matrix inversion lemma in Eq. (A.28) (with $\bar{A} = R$, $\bar{B} = H$, $\bar{C} = H^{\mathrm{T}}$, and $\bar{D} = P_k$)

$$(R + HP_kH^{\mathrm{T}})^{-1} = R^{-1} - R^{-1}H(P_k^{-1} + H^{\mathrm{T}}R^{-1}H)^{-1}H^{\mathrm{T}}R^{-1}. \tag{A.56}$$

Substitute the above into Eq. (A.54),

$$\begin{aligned}\Lambda &= \tilde{x}_k^{\mathrm{T}}[P_k^{-1} - H^{\mathrm{T}}R^{-1}H + H^{\mathrm{T}}R^{-1}H\\ &\quad\times (P_k^{-1} + H^{\mathrm{T}}R^{-1}H)^{-1}H^{\mathrm{T}}R^{-1}H]\tilde{x}_k\\ &\quad + z_k^{\mathrm{T}}[R^{-1}H(P_k^{-1} + H^{\mathrm{T}}R^{-1}H)^{-1}H^{\mathrm{T}}R^{-1}]z_k\\ &\quad + z_k^{\mathrm{T}}R^{-1}H\tilde{x}_k - z_k^{\mathrm{T}}R^{-1}H(P_k^{-1} + H^{\mathrm{T}}R^{-1}H)^{-1}\\ &\quad\times H^{\mathrm{T}}R^{-1}H\tilde{x}_k\,\tilde{x}_k^{\mathrm{T}}H^{\mathrm{T}}R^{-1}z_k - \tilde{x}_k^{\mathrm{T}}\\ &\quad\times H^{\mathrm{T}}R^{-1}H(P_k^{-1} + H^{\mathrm{T}}R^{-1}H)^{-1}H^{\mathrm{T}}R^{-1}z_k \qquad\qquad (\mathrm{A.57})\\ &= \tilde{x}_k^{\mathrm{T}}[P_k^{-1} - H^{\mathrm{T}}R^{-1}H[I - (P_k^{-1} + H^{\mathrm{T}}R^{-1}H)^{-1}\\ &\quad\times H^{\mathrm{T}}R^{-1}H]]\tilde{x}_k + z_k^{\mathrm{T}}[R^{-1}H(P_k^{-1} + H^{\mathrm{T}}R^{-1}H)^{-1}\\ &\quad\times H^{\mathrm{T}}R^{-1}]z_k + z_k^{\mathrm{T}}R^{-1}H[P_k - P_kP_k^{-1}(P_k^{-1}\\ &\quad + H^{\mathrm{T}}R^{-1}H)^{-1}H^{\mathrm{T}}R^{-1}HP_k]P_k^{-1}\tilde{x}_k + \tilde{x}_k^{\mathrm{T}}P_k^{-1}\\ &\quad\times [P_k - P_kH^{\mathrm{T}}R^{-1}H(P_k^{-1} + H^{\mathrm{T}}R^{-1}H)^{-1}]\\ &\quad\times H^{\mathrm{T}}R^{-1}z_k. \qquad\qquad (\mathrm{A.58})\end{aligned}$$

Note that by matrix inversion lemma, Eq. (A.28)

$$I - (P_k^{-1} + H^{\mathrm{T}}R^{-1}H)^{-1}H^{\mathrm{T}}R^{-1}H = [I + P_kH^{\mathrm{T}}R^{-1}H]^{-1}, \tag{A.59}$$

$$P_k - P_kP_k^{-1}(P_k^{-1} + H^{\mathrm{T}}R^{-1}H)^{-1}H^{\mathrm{T}}R^{-1}HP_k = [P_k^{-1} + P_k^{-1}P_kH^{\mathrm{T}}R^{-1}H], \tag{A.60}$$

$$P_k - P_kH^{\mathrm{T}}R^{-1}H(P_k^{-1} + H^{\mathrm{T}}R^{-1}H)^{-1} = [P_k^{-1} + H^{\mathrm{T}}R^{-1}HP_kP_k^{-1}], \tag{A.61}$$

where $\bar{A} = I$, $\bar{B} = I$, $\bar{C} = H^{\mathrm{T}}R^{-1}H$, and $\bar{D} = P_k$ in Eq. (A.59), $\bar{A} = P_k^{-1}$, $\bar{B} = P_k^{-1}$, $\bar{C} = H^{\mathrm{T}}R - 1H$, and $\bar{D} = P_k$ in Eq. (A.60), and $\bar{A} = P_k^{-1}$, $\bar{B} = H^{\mathrm{T}}R^{-1}H$, $\bar{C} = P_k^{-1}$, and $\bar{D} = P_k$ in Eq. (A.61). Substituting Eqs. (A.59) and (A.60) into Eq. (A.58) yields

$$\Lambda = \tilde{x}_k^{\mathrm{T}}[P_k^{-1} - H^{\mathrm{T}}R^{-1}H[I + P_kH^{\mathrm{T}}R^{-1}H]^{-1}]\tilde{x}_k$$

$$+z_k^{\mathrm{T}}[R^{-1}H(P_k^{-1}+H^{\mathrm{T}}R^{-1}H)^{-1}H^{\mathrm{T}}R^{-1}]z_k$$
$$+z_k^{\mathrm{T}}R^{-1}H[P_k^{-1}+P_k^{-1}P_kH^{\mathrm{T}}R^{-1}H]^{-1}P_k^{-1}\tilde{x}_k$$
$$+\tilde{x}_k^{\mathrm{T}}P_k^{-1}[P_k^{-1}+H^{\mathrm{T}}R^{-1}HP_kP_k^{-1}]^{-1}H^{\mathrm{T}}R^{-1}z_k \quad \text{(A.62)}$$

$$=\tilde{x}_k^{\mathrm{T}}P_k^{-1}[P_k-P_kH^{\mathrm{T}}R^{-1}H[I+P_kH^{\mathrm{T}}R^{-1}H]^{-1}P_k] \times P_k^{-1}\tilde{x}_k+z_k^{\mathrm{T}}[R^{-1}H(P_k^{-1}+H^{\mathrm{T}}R^{-1}H)^{-1} \times H^{\mathrm{T}}R^{-1}]z_k+z_k^{\mathrm{T}}R^{-1}H[P_k^{-1}+H^{\mathrm{T}}R^{-1}H]^{-1} \times P_k^{-1}\tilde{x}_k+\tilde{x}_k^{\mathrm{T}}P_k^{-1}[P_k^{-1}+H^{\mathrm{T}}R^{-1}H]^{-1}H^{\mathrm{T}}R^{-1}z_k. \quad \text{(A.63)}$$

From matrix inversion lemma (with $\bar{A}=P_k^{-1}$, $\bar{B}=H^{\mathrm{T}}R^{-1}H$, $\bar{C}=I$, and $\bar{D}=I$)

$$[P_k-P_kH^{\mathrm{T}}R^{-1}H[I+P_kH^{\mathrm{T}}R^{-1}H]^{-1}P_k] =[P_k^{-1}+H^{\mathrm{T}}R^{-1}H]^{-1}. \quad \text{(A.64)}$$

Substitute this into Eq. (A.63) to obtain

$$\Lambda=\tilde{x}_k^{\mathrm{T}}P_k^{-1}[P_k^{-1}+H^{\mathrm{T}}R^{-1}H]^{-1}P_k^{-1}\tilde{x}_k \times z_k^{\mathrm{T}}[R^{-1}H(P_k^{-1}+H^{\mathrm{T}}R^{-1}H)^{-1}]H^{\mathrm{T}}R^{-1}z_k +z_k^{\mathrm{T}}R^{-1}H[P_k^{-1}+H^{\mathrm{T}}R^{-1}H]^{-1}P_k^{-1}\tilde{x}_k +\tilde{x}_k^{\mathrm{T}}P_k^{-1}[P_k^{-1}+H^{\mathrm{T}}R^{-1}H]^{-1}H^{\mathrm{T}}R^{-1}z_k \quad \text{(A.65)}$$

$$=\{\tilde{x}_k^{\mathrm{T}}P_k^{-1}[P_k^{-1}+H^{\mathrm{T}}R^{-1}H]^{-1} +z_k^{\mathrm{T}}R^{-1}H[P_k^{-1}+H^{\mathrm{T}}R^{-1}H]^{-1}\} \times\{P_k^{-1}\tilde{x}_k+H^{\mathrm{T}}R^{-1}z_k\}$$
$$=[\tilde{x}_k^{\mathrm{T}}P_k^{-1}+z_k^{\mathrm{T}}R^{-1}H][P_k^{-1}+H^{\mathrm{T}}R^{-1}H]^{-1} \times[P_k^{-1}\tilde{x}_k+H^{\mathrm{T}}R^{-1}z_k]. \quad \text{(A.66)}$$

Hence the sum $\Gamma+\Lambda$ becomes

$$\Gamma+\Lambda =x_k^{\mathrm{T}}(P_k^{-1}+H^{\mathrm{T}}R^{-1}H)x_k-x_k^{\mathrm{T}}(P_k^{-1}\tilde{x}_k +H^{\mathrm{T}}R^{-1}z_k)]-(\tilde{x}_k^{\mathrm{T}}P_k^{-1}+z_k^{\mathrm{T}}R^{-1}H)x_k +[\tilde{x}_k^{\mathrm{T}}P_k^{-1}+z_k^{\mathrm{T}}R^{-1}H][P_k^{-1}+H^{\mathrm{T}}R^{-1}H]^{-1} \times[P_k^{-1}\tilde{x}_k+H^{\mathrm{T}}R^{-1}z_k] \quad \text{(A.67)}$$

$$=\{x_k^{\mathrm{T}}-[\tilde{x}_k^{\mathrm{T}}P_k^{-1}+z_k^{\mathrm{T}}R^{-1}H] \times[P_k^{-1}+H^{\mathrm{T}}R^{-1}H]^{-1}\}\{[P_k^{-1}+H^{\mathrm{T}}R^{-1}H]x_k -[P_k^{-1}\tilde{x}_k+H^{\mathrm{T}}R^{-1}z_k]\} \quad \text{(A.68)}$$

$$=\{x_k-[P_k^{-1}+H^{\mathrm{T}}R^{-1}H]^{-1} \times[P_k^{-1}\tilde{x}_k+H^{\mathrm{T}}R^{-1}z_k]\}^{\mathrm{T}}[P_k^{-1}+H^{\mathrm{T}}R^{-1}H] \times\{x_k-[P_k^{-1}+H^{\mathrm{T}}R^{-1}H]^{-1} \times[P_k^{-1}\tilde{x}_k+H^{\mathrm{T}}R^{-1}z_k]\}. \quad \text{(A.69)}$$

Substitute Eq. (A.69) into Eq. (A.55) to obtain

$$\mu_{X(k)\times Z(k)}(x_k,z_k) =\exp\{-\tfrac{1}{2}[(x_k-\tilde{x}_k^{+})^{\mathrm{T}}(P_k^{+})^{-1}(x_k-\tilde{x}_k^{+})^{\mathrm{T}}]\} \times\exp\{-\tfrac{1}{2}[(z_k-H\tilde{x}_k)^{\mathrm{T}}(R+HP_kH^{\mathrm{T}})^{-1} \times(z_k-H\tilde{x}_k)]\}, \quad \text{(A.70)}$$

where

$$\tilde{x}_k^{+}=P_k^{+}P_k^{-1}\tilde{x}_k+P_k^{+}H^{\mathrm{T}}R^{-1}z_k, \quad \text{(A.71)}$$
$$P_k^{+}=[P_k^{-1}+H^{\mathrm{T}}R^{-1}H]^{-1}. \quad \text{(A.72)}$$

Recall that the measurement update is performed by normalizing Eq. (A.70), i.e.

$$\mu_{X^{+}(k)|Z(k)=z_k}(X_k^{+})=\mathcal{N}(\mu_{X(k)\times Z(k)}(x_k,z_k)). \quad \text{(A.73)}$$

Since the last term in Eq. (A.70) is a simple constant for a given z_k and $\tilde{x}_k$ this reduces to

$$\mu_{X^{+}(k)|Z(k)=z_m}(X_k^{+}) =\exp\{-\tfrac{1}{2}[(x_k-\tilde{x}_k^{+})^{\mathrm{T}}(P_k^{+})^{-1}(x_k-\tilde{x}_k^{+})^{\mathrm{T}}]\}, \quad \text{(A.74)}$$

where

$$\tilde{x}_k^{+}=P_k^{+}P_k^{-1}\tilde{x}_k+P_k^{+}H^{\mathrm{T}}R^{-1}z_k, \quad \text{(A.75)}$$
$$P_k^{+}=[P_k^{-1}+H^{\mathrm{T}}R^{-1}H]^{-1}. \quad \text{(A.76)}$$

Notice that Eq. (A.74) is a normalized Gaussian membership function and Eqs. (A.75) and (A.76) are easily recognized as being analogous to the familiar measurement update equations for the Kalman filter [15].

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