Stability of a One-Dimensional Discrete-Time Asynchronous Swarm

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Abstract— **In this article we consider a discrete time one-dimensional asynchronous swarm. First, we describe the mathematical model for motions of the swarm members. Then, we analyze the stability properties of that model. The stability concept that we consider, which matches exactly with stability of equilibria in control theory, characterizes stability of a particular position (relative arrangement) of the swarm members, that we call the *comfortable position* (with *comfortable intermember distance*). Our stability analysis employs some results on contractive mappings from the parallel and distributed computation literature.**

I. Introduction

The area of swarm stability analysis is an active research area that has become more important due to its potential use in characterizing and analyzing mechanisms for cooperative control for groups of autonomous vehicles. Important past work includes the work done by mathematical biologists [1], [2], [3], where they consider models of the density of the swarm and study its properties. In [4] Jin et al. studied the stability properties of one-dimensional and two-dimensional synchronized swarms. Note that the stability of one dimensional swarms is similar to the concept of "platoon" stability in automated highway systems and there has been a significant work in that area (see, for example, [5], [6], [7], [8]). On the other hand, [9] is, to best of our knowledge, one of the first stability results for asynchronous methods. There they consider a "linear" swarm model and prove sufficient conditions for the asynchronous convergence of the swarm to a synchronously achievable configuration. Although their method is asynchronous, they do not have time delays in the system. The stability of totally asynchronous swarm models (i.e., asynchronous swarm models with time delay) was, to best of our knowledge, first considered by Liu et al. in [10], [11]. In [10] they analyze the stability (cohesiveness) of one-dimensional asynchronous swarm model, whereas in [11] the stability of one-dimensional mobile swarm model is considered.

In this work, we use the representation of a single swarm member from [10], [11]; however, we consider a different mathematical model for the intermember interactions and motions in a swarm. We prove stability of the comfortable position for the new model using different mathematical tools for analysis. Namely, we use some earlier results developed for computer networks in [12]. First, we prove the stability in case of synchronism with no delays. Then, we use this result to prove the stability under total asynchronism (with included delays).

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II. The Swarm Model

In this section we introduce the swarm model that we use in this article. First, we describe the model of a single swarm member. Then, we present the one-dimensional swarm model (i.e., when many swarm members are arranged next to each other on a line).

A. Single Swarm Member Model

The single swarm model described in this section is taken from [10], [11]. We present it here for convenience. The single swarm member model that we consider is shown in Figure 1. As seen in the figure, it has a *driving device*

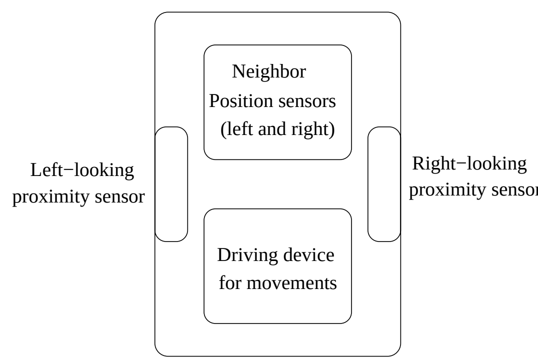

Fig. 1. Single swarm member.

for performing the movements and a *neighbor position sensors* for sensing the position of the adjacent (left and right) neighbors. It is assumed that there is no restriction on the range on these sensors. In other words, we assume that they can provide the accurate position of the neighbor even if the neighbor is far away. Each swarm member also has two proximity sensors on both sides (left and right). These sensors have sensing range of $\epsilon > 0$ and can sense *instantaneously* in this proximity. Therefore, if another swarm member reaches an ϵ distance from it, then this will be *instantaneously* known by both of the members. However, if the neighbors of the swarm member are out of the range of the proximity sensor, then it will return an infinite value (i.e., $-\infty$ for the left sensor and $+\infty$ for the right sensor) or some large number that will be ignored by the swarm member. The use of this sensor is to avoid collisions with the other members in the swarm.

In the next section we describe the model of a swarm (collection) of members described in this section arranged on a line.

B. One-Dimensional Swarm Model

Consider a discrete time one-dimensional swarm described by the model

$$\begin{aligned} x_1(k+1) &= x_1(k), \forall k \\ x_i(k+1) &= \max\big\{x_{i-1}(k)+\epsilon, \min\big\{x_i(k) - g\big(x_i(k) \\ &\quad -\tfrac{x_{i-1}(\tau^i_{i-1}(k))+x_{i+1}(\tau^i_{i+1}(k))}{2}\big), \\ &\quad x_{i+1}(k)-\epsilon\big\}\big\}, \forall k \in \mathcal{K}^i, i=2,\ldots,N-1 \\ x_N(k+1) &= \max\big\{x_{N-1}(k)+\epsilon, x_N(k) - g\big(x_N(k) \\ &\quad -x_{N-1}(\tau^N_{N-1}(k)) - d\big)\big\}, \forall k \in \mathcal{K}^N, \end{aligned} \tag{1}$$

where $x_i(k), i = 1, \ldots, N$, represents the position of member i at time k and $\mathcal{K}^i \subseteq \mathcal{K} = \{1, 2, \ldots\}$ is the set of time instants at which member i updates its position. At the other time instants member i is stationary. In other words, we have

$$x_i(k+1) = x_i(k), \forall k \notin \mathcal{K}^i \text{ and } i = 2, \ldots, N. \tag{2}$$

Note that the first member of the swarm is always stationary at position $x_1(0)$. The other members (except member N), on the other hand, try to move to the position which their current information tells them is the middle of their adjacent neighbors. In other words, they try to move to the position $c_i(k)$ defined as

$$c_i(k) = \frac{x_{i-1}(\tau^i_{i-1}(k)) + x_{i+1}(\tau^i_{i+1}(k))}{2}, i = 2, \ldots, N-1,$$

where $\tau^i_j, j = i-1, i+1$, is used to represent the time index at which member i obtained position information of its neighbor j. Of course due to the delays $c_i(k)$ may not be the midpoint between members $i-1$ and $i+1$ at time k. The last member (member N), on the other hand, tries to move to

$$c_N(k) = x_{N-1}(\tau^N_{N-1}(k)) + d,$$

what it perceives to be a distance d from its left neighbor. The constant d represents the *comfortable intermember distance*. Note that, in contrast to the work in [10], [11], only the N^{th} member of the swarm knows (or decides) the value of d. It is assumed that $d \gg \epsilon$.

The elements of $\mathcal{K}$ (and therefore of $\mathcal{K}^i$) should be viewed as indices of the sequence of physical times at which the updates occur (similar to the times of events in discrete event systems), not as actual times. In other words, they are integers that can be mapped to actual times. The sets $\mathcal{K}^i$ are independent from each other for different i. However, it is possible to have $\mathcal{K}^i \cap \mathcal{K}^j \neq \emptyset$ for $i \neq j$ (i.e., two or more members move simultaneously). Note that $\tau^i_j(k)$ satisfies $0 \leq \tau^i_j(k) \leq k$ for $k \in \mathcal{K}^i$, where $\tau^i_j(k) = 0$ means that member i did not obtain any position information about member j so far (it still has the initial position information), whereas $\tau^i_j(k) = k$ means that it has the current position information of member j. The constant ϵ is the range of the proximity sensors as discussed in the preceding section.

The function g describes the attractive and repelling relationships between a swarm member and its adjacent neighbors. It determines the step size that a member will take toward the middle of its neighbors (if it is not already there). We assume that

$$\begin{aligned} \underline{\alpha} y(t) \leq g(y(t)) \leq \bar{\alpha} y(t), &\text{ if } y(t) \geq 0 \\ \bar{\alpha} y(t) \leq g(y(t)) \leq \underline{\alpha} y(t), &\text{ if } y(t) < 0, \end{aligned} \tag{3}$$

where $\underline{\alpha}$ and $\bar{\alpha}$ are two constants satisfying

$$0 < \underline{\alpha} < \bar{\alpha} < 1.$$

Figure 2 shows the plot of one such g. In the figure we also plotted $\underline{\alpha} y(t)$ and $\bar{\alpha} y(t)$ for $\underline{\alpha} = 0.1$ and $\bar{\alpha} = 0.9$.

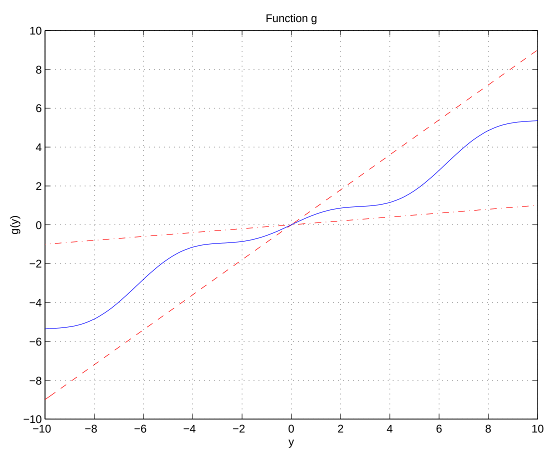

Fig. 2. The g function.

Notice that the model in Eq. (1) is in a sense a discrete event model which does not allow for collisions between the swarm members. This is because if during movement member i suddenly finds itself within an ϵ range of one (or both) of its neighbors, it will restrain its movement by that neighbor according to Eq. (1).

We will at times use the notation $x(k) = [x_1(k), \ldots, x_N(k)]^\top$ to represent the position at time k of all the members of the swarm. Define the swarm comfortable position as

$$x^c = [x_1(0), x_1(0) + d, \ldots, x_1(0) + (N-1)d]^\top.$$

In this article we consider the stability of this position by considering the motions of the swarm members when they are initialized at positions different from x^c. We will consider two cases: synchronous operation with no delays and totally asynchronous operation. These are described in the following two assumptions.

Assumption 1: (Synchronism, No Delays) The sets $\mathcal{K}^i$ and the times $\tau^i_j(k)$ satisfy $\mathcal{K}^i = \mathcal{K}$ for all i and $\tau^i_j(k) = k$ for all i and $j = i-1, i+1$.

This assumption says that all the swarm members will move at the same time instants. Moreover, every member will always have the current position information of its adjacent neighbors.

The next assumption, on the other hand, says that the members can move at totally independent time instants and that the "delay" between two measurements performed by a member can become arbitrarily large. However, there always will be next time when the member will perform a measurement.

Assumption 2: (Total Asynchronism) The sets $\mathcal{K}^i$ are infinite, and if $\{k_\ell\}$ is a sequence of elements of $\mathcal{K}^i$ that tends to infinity, then $\lim_{\ell\to\infty} \tau_j^i(k_\ell) = \infty$ for every j.

Now we have the following preliminary result. We state it here, because it will be used in the next section.

Lemma 1: For the swarm described in Eq. (1) given any $x(0)$, there exists a constant $\bar{b} = \bar{b}(x(0))$ such that $x_i(k) \leq \bar{b}$, for all k and all $i, 1 \leq i \leq N$.

Proof: We prove this via contradiction. Assume that $x_i(k) \to \infty$ for some $i, 1 \leq i \leq N$. This implies that $x_j(k) \to \infty$ for all $j \geq i$. We will show that it must be the case that $x_{i-1}(k) \to \infty$. Assume the contrary. Then we have $x_i(k) - x_{i-1}(k) \to \infty$, whereas $x_{i-1}(k) - x_{i-2}(k) < b$ for some b. However, there is always a time $k^{i-1} \in \mathcal{K}^{i-1}$ at which member $i-1$ performs position sensing of its neighbors and since at some time $x_i(k) - x_{i-1}(k) \gg x_{i-1}(k) - x_{i-2}(k)$, it moves to the right. Repeating the argument for each time instant, we obtain $x_{i-1}(k) \to \infty$. Continuing this way it can be shown that $x_i(k) \to \infty$ for all $i \neq 1$. Moreover, since x_1 is constant and $x_2(k) - x_1(k) \to \infty$ we have all $x_i(k) - x_{i-1}(k) \to \infty, i = 2, \ldots, N$. To see this assume that $x_2(k) - x_1(k) \to \infty$, whereas $x_3(k) - x_2(k) < b$ for some b. Then, there exists always a time $k^2 \in \mathcal{K}^2$ at which member 2 performs a position sensing of its neighbors and it moves to the left. Therefore, it must be the case that $x_3(k) - x_2(k) \to \infty$. Repeating the argument for the other members we arrive at the conclusion that it should hold for all i. This leads to a contradiction since there is always a time $k^N \in \mathcal{K}^N$ at which member N performs position sensing of its left neighbor. From the definition of the model if $x_N(k) - x_{N-1}(k) > d$ the N^{th} member will move to the left. In other words, $x_N(k) - x_{N-1}(k)$ cannot diverge. Then, there is always a time $k^{N-1} \in \mathcal{K}^N, k^{N-1} > k^N$ at which member $N-1$ performs position sensing of the neighbors, and since $x_{N-1}(k) - x_{N-2}(k) > x_N(k) - x_{N-1}(k)$ it moves to left. Therefore, $x_{N-1}(k) - x_{N-2}(k)$ also cannot diverge. Continuing with similar reasoning one can show that all $x_i(k) - x_{i-1}(k)$ are bounded implying the result. ∎

This result is important, because it basically says that for the given swarm model unboundednes of the swarm member positions and intermember distances (the dissolution of the swarm) will not occur. Therefore, the main question to be answered is whether the swarm member positions $x(k)$ will have periodic solutions or will converge to some constant. In the next section we will analyze the system in the case of synchronism with no delays. This will be used later in the proof of our main result.

III. The System Under Total Synchronism

In this section we will assume that Assumption 1 holds (i.e., all the members move at the same time and they always have the current position information of the neighbors) and analyze the stability properties of the system.

Now we have the following preliminary result.

Lemma 2: For the system in Eq. (1) assume that Assumption 1 holds (i.e., we have synchronism with no delays). If $x(k) \to \bar{x}$ as $k \to \infty$, where $\bar{x}$ is a constant vector, then $\bar{x} = x^c$.

Proof: First of all, note that the intermember distances on all the states that the system can converge to are such that $\bar{x}_i - \bar{x}_{i-1} > \epsilon$ for all i (i.e., it is impossible for the states to converge to positions that are very close to each other). To prove this, we assume that $\bar{x}_i - \bar{x}_{i-1} = \epsilon$ for some i and $\bar{x}_j - \bar{x}_{j-1} > \epsilon$ for all $j \neq i$ and seek to show a contradiction. In that case,

$$\bar{x}_{i+1} - \bar{x}_i > \epsilon$$

so

$$\bar{x}_i - \frac{\bar{x}_{i-1} + \bar{x}_{i+1}}{2} < 0$$

and we have from model constraints in Eq. (1) that

$$\bar{x}_{i-1} + \epsilon < \bar{x}_i - g\big(\bar{x}_i - \frac{\bar{x}_{i-1} + \bar{x}_{i+1}}{2}\big) < \bar{x}_{i+1} - \epsilon.$$

From Eq. (1) this implies that at the next time instant $k^i \in \mathcal{K}^i$ member i will move to the right toward member $i+1$. Therefore, it must be the case that $\bar{x}_{i+1} - \bar{x}_i = \epsilon$ since otherwise $\bar{x}_i - \bar{x}_{i-1} = \epsilon$ also cannot hold. Continuing this way one can prove that all intermember distances must be equal to ϵ. However, in that case, since $d \gg \epsilon$, from last equality in Eq. (1) we have

$$\bar{x}_N - g(\epsilon - d) > \bar{x}_{N-1} + \epsilon$$

and this implies that on the next time instant $k^N \in \mathcal{K}^N$ member N will move to the right. Therefore, no intermember distance can converge to ϵ. For this reason, to find $\bar{x}$ we can drop the min and max and consider only the middle terms in Eq. (1).

Since $x(k) \to \bar{x}$ as $t \to \infty$ it should be the case that ultimately

$$\begin{aligned} \bar{x}_1 &= \bar{x}_1 \\ \bar{x}_i &= \bar{x}_i - g\big(\bar{x}_i - \frac{\bar{x}_{i-1} + \bar{x}_{i+1}}{2}\big), i = 1, \ldots, N-1 \\ \bar{x}_N &= \bar{x}_N - g\big(\bar{x}_N - \bar{x}_{N-1} - d\big), \end{aligned}$$

from which we obtain

$$\begin{aligned} \bar{x}_1 &= x_1^c \\ 2\bar{x}_i &= \bar{x}_{i-1} + \bar{x}_{i+1}, i = 1, \ldots, N-1 \\ \bar{x}_N &= \bar{x}_{N-1} + d. \end{aligned} \tag{4}$$

Solving the second equation for $\bar{x}_{N-1}$ we have

$$2\bar{x}_{N-1} = \bar{x}_{N-2} + \bar{x}_N$$

from which we obtain

$$\bar{x}_{N-1} = \bar{x}_{N-2} + d.$$

Continuing this way, we obtain

$$\bar{x}_i = \bar{x}_{i-1} + d, \forall i = 1, \ldots, N-1.$$

Then since the first member is stationary we have $\bar{x}_1 = x_1(t) = x_1(0) = x_1^c$ and this proves the result. ■

This lemma basically says that x^c is the unique *fixed point* or *equilibrium point* of the system described by Eq. (1). In this article we analyze the stability of this fixed point which corresponds to the arrangement with comfortable intermember distance.

Lemma 3: Assume that $x_i(0) - x_{i-1}(0) > \epsilon$ for all $i = 2, \ldots, N$. Moreover, assume that Assumption 1 holds (i.e., we have synchronism with no delays). Then, $x_i(k) - x_{i-1}(k) > \epsilon$ for all $i = 2, \ldots, N$, and for all k.

Proof: We will prove this by induction. By assumption for $k = 0$ we have $x_i(0) - x_{i-1}(0) > \epsilon$ for all $i = 2, \ldots, N$. Assume that for some k we have $x_i(k) - x_{i-1}(k) > \epsilon$ for all $i = 2, \ldots, N$. Then we have

$$\frac{x_{i-1}(t) + x_{i-2}(t)}{2} < \frac{x_i(t) + x_{i-1}(t)}{2} - \epsilon. \tag{5}$$

On the other hand, from Eq. (4) we have

$$\begin{aligned} x_i(k+1) &= x_i(k) - \alpha_i\big(x_i(k) - \frac{x_{i-1}(k) + x_{i-2}(k)}{2}\big) \\ &= (1-\alpha_i)x_i(k) + \alpha_i\big(\frac{x_{i-1}(k) + x_{i-2}(t)}{2}\big), \end{aligned}$$

where $\underline{\alpha} < \alpha_i < \bar{\alpha}$. Therefore, as shown in Figure 3, we have

if $\quad x_i(k) < \frac{x_{i-1}(k)+x_{i-2}(k)}{2}$
then $\quad x_i(k) < x_i(k+1) < \frac{x_{i-1}(k)+x_{i-2}(k)}{2}$
and
if $\quad x_i(k) > \frac{x_{i-1}(k)+x_{i-2}(k)}{2}$
then $\quad x_i(k) > x_i(k+1) > \frac{x_{i-1}(k)+x_{i-2}(k)}{2}$.

Then, Eq. (5) implies that $x_i(k+1) - x_{i-1}(k+1) > \epsilon$ and

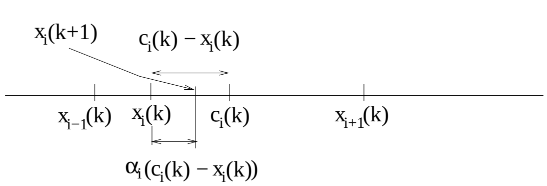

Fig. 3. Step of a swarm member.

this completes the proof. ■

This lemma implies that for the synchronous case with no delays, provided that initially the members are sufficiently apart from each other, the proximity sensors will not be used and that we can drop the min and max operations in Eq. (1) and the system can be represented as

$$\begin{aligned} x_1(k+1) &= x_1(k) \\ x_i(k+1) &= x_i(k) - g\big(x_i(k) - \frac{x_{i-1}(k) + x_{i+1}(k)}{2}\big), \\ x_N(k+1) &= x_N(k) + g\big(x_N(k) - x_{N-1}(k) - d\big). \end{aligned}$$

Define the following change of coordinates

$$\begin{aligned} e_1(k) &= x_1(k) - x_1^c \\ e_i(k) &= x_i(k) - (x_{i-1}(k) + d), i = 1, \ldots, N. \end{aligned}$$

Then, one obtains the following representation of the system

$$\begin{aligned} e_1(k+1) &= e_1(k) = 0, \\ e_2(k+1) &= e_2(k) - g\big(\frac{e_2(k) - e_3(k)}{2}\big), \\ e_i(k+1) &= e_i(k) - g\big(\frac{e_i(k) - e_{i+1}(k)}{2}\big) \\ &\quad + g\big(\frac{e_{i-1}(k) - e_i(k)}{2}\big), i = 3, \ldots, N-1, \\ e_N(k+1) &= e_N(k) - g\big(e_N(k)\big) + g\big(\frac{e_{N-1}(k) - e_N(k)}{2}\big). \end{aligned}$$

Noting that it is possible to write the g function as

$$g(y(k)) = \alpha(k)y(k),$$

where

$$0 < \underline{\alpha} \leq \alpha(k) \leq \bar{\alpha} < 1,$$

we can represent the system with

$$\begin{aligned} e_2(k+1) &= \big(1 - \frac{\alpha_2(k)}{2}\big)e_2(k) + \frac{\alpha_2(k)}{2}e_3(t), \\ e_i(k+1) &= \big(1 - \frac{\alpha_i(k)}{2} - \frac{\alpha_{i-1}(k)}{2}\big)e_i(k) \\ &\quad + \frac{\alpha_{i-1}(k)}{2}e_{i-1}(k) \\ &\quad + \frac{\alpha_i(k)}{2}e_{i+1}(k), i = 3, \ldots, N-1, \\ e_N(k+1) &= \big(1 - \alpha_N(k) - \frac{\alpha_{N-1}(k)}{2}\big)e_N(k) \\ &\quad + \frac{\alpha_{N-1}(k)}{2}e_{N-1}(k), \end{aligned}$$

where we dropped $e_1(k)$ since it is zero for all k. In other words, our system is, in a sense, a linear time varying system of the form

$$e(k+1) = A(k)e(k),$$

where $e(k) = [e_2(k), \ldots, e_N(k)]^\top$ and $A(k)$ is a symmetric tridiagonal matrix with diagnal

$$\Big\{\big(1 - \frac{\alpha_2(k)}{2}\big), \big(1 - \frac{\alpha_3(k)}{2} - \frac{\alpha_2(k)}{2}\big), \ldots,$$

$$\big(1 - \frac{\alpha_{N-1}(k)}{2} - \frac{\alpha_{N-2}(k)}{2}\big), \big(1 - \alpha_N(k) - \frac{\alpha_{N-1}(k)}{2}\big)\Big\}$$

and offdiagonal

$$\Big\{\frac{\alpha_2(k)}{2}, \ldots, \frac{\alpha_{N-1}(k)}{2}\Big\}.$$

Now we present the following lemma that will be used later.

Lemma 4: The spectrum of the matrix $A(k)$, $\rho(A(k))$ satisfies

$$\rho(A(k)) \leq 1$$

for all k.

Proof: Note that for the given $A(k)$ we have

$$\|A(k)\|_1 = \|A(k)\|_\infty = 1$$

for all k. On the other hand, for any given matrix $A(k)$ it is well known that the two norm satisfies

$$\|A(k)\|_2 \leq \|A(k)\|_1 \|A(k)\|_\infty.$$

Hence, since we have

$$\rho(A(k)) = \|A(k)\|_2,$$

we obtain

$$\rho(A(k)) \leq 1$$

for all k, which completes the proof. ■

This lemma basically says that the eigenvalues of $A(k)$ (which are all real numbers since $A(k)$ is symmetric) lie on the unit disc for each k. However, this result is not satisfactory and we need to prove that all of the eigenvalues of $A(k)$ lie within the unit circle for each k. This is done with the help of the next lemma.

Lemma 5: Let $\underline{\alpha} \leq \alpha_i(k) = \alpha_i \leq \bar{\alpha}$ for all k and $i = 2, \ldots, N$ (i.e., the α_i's in the matrix A are all constants). Then,

$$\rho(A(k)) = \rho(A) < 1,$$

and we have $e(k) \to 0$ as $k \to \infty$.

Proof: To prove the assertion, note that $A(k)$ is a symmetric matrix. Therefore, there exists a unitary transformation P (i.e., $P^{-1} = P^\top$) such that $\bar{A} = PAP^\top$, where $\bar{A} = diag\{\bar{a}_2, \ldots, \bar{a}_N\}$. For the sake of contradiction assume that $\rho(A) = 1$. Then, it must be the case that $\bar{a}_i = 1$ for some $i, 2 \leq i \leq N$. Define the transformation $\bar{e} = Pe$. Then the system can be described as

$$\bar{e}(k+1) = \bar{A}\bar{e}(k).$$

Since $\bar{A}$ is diagonal and $\bar{a}_i = 1$ we have $\bar{e}_i(k) = \bar{e}_i(0)$ for all k, whereas $\bar{e}_j(k) \to 0$ as $k \to \infty$ for all $j \neq i$. This, on the other hand, implies that $e(k) \to P_i \bar{e}_i(0) = e^c$ as $k \to \infty$, where P_i is the i^{th} column of P. Depending on the value of $\bar{e}_i(0)$, the value of e^c can be any number. However, this contradicts the result of Lemma 2. Therefore, $\bar{a}_i < 1$ for all $i = 2, \ldots, N$, and this implies that $\rho(A) < 1$. ■

Since in the above lemma $\alpha = [\alpha_2, \ldots, \alpha_N]^\top$ was chosen arbitrary, the result holds for all α such that $\underline{\alpha} \leq \alpha_i \leq \bar{\alpha}$. Hence, we have

$$\rho(A(k)) < 1$$

for each k. Before proceeding define

$$\bar{\rho} = \sup_{\underline{\alpha} \leq \alpha_i \leq \bar{\alpha}, i=2\ldots N} \{\rho(A)\}.$$

Then, from the above result we have

$$\bar{\rho} < 1.$$

IV. The Main Result

In this section we return to the totally asynchronous case. In other words, we assume that Assumption 2 holds. To prove its stability we will use the result from the synchronous case and a result from [12]. For convenience we present this result here.

Consider the function $f : X \to X$, where $X = X_1 \times \ldots, \times X_n$, and $x = [x_1, \ldots, x_n]^\top$ with $x_i \in X_i$. The function f is composed of functions $f_i : X \to X_i$ in the form $f = [f_1, \ldots, f_n]^\top$ for all $x \in X$. Consider the problem of finding the point x^* such that

$$x^* = f(x^*)$$

using an asynchronous algorithm. In other words, use an algorithm in which

$$x_i(k+1) = f_i\big(x_1(\tau_1^i(k)), \ldots, x_n(\tau_n^i(k))\big), \forall t \in \mathcal{K}^i,$$

where $\tau_j^i(k)$ are times satisfying

$$0 \leq \tau_j^i(k) \leq k, \forall k \in \mathcal{K}.$$

For all the other times $k \notin \mathcal{K}^i$, x_i is left unchanged. In other words, we have

$$x_i(k+1) = x_i(k), \forall k \notin \mathcal{K}^i.$$

Consider the following assumption.

Assumption 3: There is a sequence of nonempty sets $\{X(k)\}$ with

$$\cdots \subset X(k+1) \subset X(k) \subset \cdots \subset X,$$

satisfying the following two conditions:

1. *Synchronous Convergence Condition (SCC):* We have

$$f(x) \in X(k+1), \forall k \text{ and } x \in X(k).$$

Furthermore, if $\{y_k\}$ is a sequence such that $y_k \in X(k)$ for every k, then every limit point of $\{y_k\}$ is a fixed point of f.

2. *Box Condition (BC):* For every k, there exist sets $X_i(k) \subset X_i$ such that

$$X(k) = X_1(k) \times X_2(k) \times \ldots X_n(k).$$

Then we have the following result.

Theorem 1: Asynchronous Convergence Theorem [12]: If the synchronous convergence condition and box condition of Assumption 3 hold, and the initial solution estimate $x(0) = [x_1(0), \ldots, x_n(0)]^\top$ belongs to the set $X(0)$, then every limit point of $\{x(k)\}$ is a fixed point of f.

This is a powerful result that can be applied to many different problems. The main idea behind its proof is that if there is a time k_1 such that $x_j(\tau_j^i(k_1)) \in X_j(k)$ for all j and all i, then the SCC and the BC conditions above guarantee that $x(k_1+1) \in X(k+1)$. Then, $x(k) \in X(k+1)$ for all $k \geq k_1$ and due to the total asynchronism assumption there will be always another time $k_2 > k_1$ such that $x_j(\tau_j^i(k_2)) \in X_j(k+1)$ for all j and all i. Since initially we have $x_j(\tau_j^i(0)) = x_j(0) \in X_j(0)$, we can use the above arguments in an induction.

Now we state our main result.

Theorem 2: For the N-member swarm modeled in Eq. (1) with $g(\cdot)$ as given in Eq. (3), if Assumption (2) holds and $x_{i+1}(0) - x_i(0) > \epsilon, i = 1, \ldots, N-1$, then the swarm member positions will converge asymptotically to the comfortable position x^c.

Proof: In order to prove this result we once again consider the synchronous case. Recall that for this case the system can be described by

$$e(k+1) = A(k)e(k).$$

In the previous section it was shown that for the synchronous case we have $\lambda(A(k)) \leq \bar{\rho} < 1$ for all k and that $e(k) \to 0$ as $k \to \infty$ (i.e., the position with comfortable intermember distance x^c). This implies that $A(k)$ is a maximum norm contraction mapping for all k. Define the sets

$$E(k) = \{e \in \mathbb{R}^{N-1} : \|e\|_\infty \leq \bar{\rho}^k \|e(0)\|_\infty\}.$$

Then since $A(k)$ is a maximum norm contraction mapping for all k we have $e(k) \in E(k)$ for all k and

$$\ldots \subset E(k+1) \subset E(k) \subset \ldots \subset E = \mathbb{R}^{N-1}.$$

Moreover, each $E(k)$ can be expressed as

$$E(k) = E_2(k) \times E_3(k) \times \ldots E_N(k).$$

Since the position with comfortable intermember distance $e = 0$ (i.e., $x = x^c$) is the unique fixed point of the system and the synchronous swarm converges to it, it is implied that Assumption 3 above is satisfied. Applying the Asynchronous Convergence Theorem we obtain the result. ∎

This result is important because it says that the stability of the system will be preserved (i.e., the system will converge to the comfortable distance) even though we have totally asynchronous motions. Note that the fact that in the asynchronous case the min and max operations are preserved does not change the result since the stability properties of the synchronous system is preserved even with them present in the model. In fact, having them is, in a sense, beneficial because they also serve as another neighbor position sensing by the members that come to an ϵ distance from each other and this provides more accurate neighbor position information.

A direct consequence of Theorem 2 is the stability of swarm in which one member in the middle is stationary, whereas all the other middle members try to move as above and both of the edge members try to move to a distance d from their neighbors. In other words, suppose the swarm is described by

$$\begin{aligned}
x_1(k+1) &= \min\big\{x_1(k) - g\big(x_1(k) + d - x_2(\tau_2^1(k))\big), \\
&\qquad x_2(k) - \epsilon\big\} \forall k \in \mathcal{K}^1, \\
x_j(k+1) &= x_j(k), \forall k \text{ and for some } j, 1 \leq j \leq N \\
x_i(k+1) &= \max\big\{x_{i-1}(k) + \epsilon, \min\big\{x_i(k) - g\big(x_i(k) \\
&\qquad - \frac{x_{i-1}(\tau_{i-1}^i(k)) + x_{i+1}(\tau_{i+1}^i(k))}{2}\big), \\
&\qquad x_{i+1}(k) - \epsilon\big\}\big\}, \forall k \in \mathcal{K}^i, \\
&\qquad i = 2, \ldots, N-1, i \neq j, \\
x_N(k+1) &= \max\big\{x_{N-1}(k) + \epsilon, x_N(k) - g\big(x_N(k) \\
&\qquad - x_{N-1}(\tau_{N-1}^N(k)) - d\big)\big\}, \forall k \in \mathcal{K}^N. \qquad (6)
\end{aligned}$$

In this case we have the following corollary as a direct consequence of Theorem 2.

Corollary 1: For the N-member swarm modeled in Eq. (6) with $g(\cdot)$ as given in Eq. (3), if Assumption (2) holds and $x_{i+1}(0) - x_i(0) > \epsilon, i = 1, \ldots, N-1$, then the swarm member positions will converge asymptotically to x^c, where x^c is defined such that $x_j^c = x_j(0)$ and $x_i^c = x_j(0) + (i-j)d$, for all $i \neq j$.

The importance of this result is for systems in which the "leader" of the swarm is not the first (or the last) member, but a member in the middle.

V. Conclusion

In this article we present one-dimensional asynchronous swarm model and analyze its stability. We show that for our model we have asymptotic convergence of the positions of the swarm members to the comfortable position despite the presence of delays and asynchronism.

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