Third-order small-perturbation method for scattering from dielectric rough surfaces

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The small-perturbation method (SPM) for rough surface scattering, originally derived by Rice [Commun. Pure Appl. Math. **4**, 361 (1951)], has been applied extensively to problems in optics, remote sensing, and propagation. Typical uses of the theory involve only the first- or second-order scattered fields in surface height, owing to increasing complexity of the SPM equations as order increases. The SPM equations are solved in a systematic manner that permits third order in surface-height terms to be determined apparently for the first time for scattering from a dielectric surface rough in two directions. Sample results for both periodic and nonperiodic surfaces show that third-order field terms can contribute to fourth-order scattered power and also to a third-order specular-reflection coefficient correction for surfaces with nonvanishing bispectra. The latter case is of particular interest in passive remote sensing of the ocean, since these third-order terms contribute to the first prediction of a first azimuthal harmonic of ocean brightness temperatures. © 1999 Optical Society of America [S0740-3232(99)01911-0]

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1. INTRODUCTION

Scattering from rough surfaces, both deterministic or stochastic and periodic or nonperiodic, is of interest in many fields of applied optics and electromagnetics. Although a general solution does not exist, numerous approximate analytical methods have been developed and have been successful for analyzing scattering from certain classes of surfaces. One of the most widely used methods is the small-perturbation method (SPM) originally developed in Ref. 1. The SPM approximation requires that surface heights be small in terms of the electromagnetic wavelength, and a perturbation series in surface height is used to determine scattered fields at a specified order in surface height. The method has been studied and applied extensively[2–18] to problems in optics, remote sensing, and propagation and yields the Bragg scatter phenomenon of rough-surface scattering when only first-order terms are considered. Scattered fields up to second order in surface height were considered in the original paper,[1] but solutions increase in complexity as order increases, so third-order terms for a general three-dimensional dielectric surface scattering problem have apparently not been previously presented. Second-order terms have only rarely been applied, and then primarily only to study the SPM correction to the flat-surface reflection coefficient[5,13,14] or cross-polarized backscatter from a Gaussian random process surface.[2]

Although third-order scattered fields from the SPM are expected to be complex and to require a numerical integration for their evaluation, recent developments in the study of thermal emission from rough surfaces have motivated new interest in higher-order SPM solutions. In particular, consideration of thermal emission from the ocean surface has shown some limitations of a model based on the second-order SPM alone.[13,16] The problems stem from the fact that second-order emission predictions are expressed entirely in terms of the surface power spectrum, which by definition is a symmetric quantity with respect to a 180° shift in azimuth. However, measurements of ocean surface emission clearly show first-harmonic azimuthal variations (i.e., having variations with respect to a 180° shift in azimuth). A third-order description of the surface-height profile is necessary to capture these properties, and consequently a third-order SPM model for surface emission is necessary if a consistent first-harmonic prediction is to be obtained. Further motivation for use of the SPM in studying surface emission comes from Ref. 15, which shows that the SPM produces an expansion in surface slope, not surface height, for the total power reflected or emitted from a rough surface.

In this paper, SPM scattered fields are derived to third order in surface height, and expressions for scattered and transmitted powers are developed for deterministic and stochastic, periodic and nonperiodic surfaces. The original formulation of Ref. 1 is followed, in which the Rayleigh hypothesis is applied to determine reflected and transmitted Floquet-mode amplitudes scattered from a periodic surface. Scattering cross sections per unit area for a nonperiodic surface are obtained following Refs. 1 and 2 in the limit as the surface periods approach infinity. Although alternative SPM studies have begun with the nonperiodic-surface case and have avoided the Rayleigh hypothesis,[7,9] in all comparisons made to date it has been shown that these approaches yield identical results. Use of the Rayleigh hypothesis can be qualitatively justified by realizing that it should hold under the same conditions for which the SPM is applicable, so an SPM derivation using the Rayleigh hypothesis should be valid. Further discussions on use of Rayleigh hypothesis can be found in Ref. 12.

Use of the Rayleigh hypothesis with a periodic surface simplifies the analysis considerably, and a systematic procedure for solving the SPM equations at any order results. The resulting equations can be implemented on a computer to permit numerical solution of the SPM equations to arbitrary order, but this procedure does not yield the insight that is available from an analytical solution. Analytical results are presented for scattered and transmitted fields to third order, and the systematic nature of the procedure permits higher-order terms to be found without extensive additional effort. Although SPM field solutions to second order are available in the literature, solutions at each order are revisited to illustrate the development and application of the new systematic solution.

Section 2 presents the basic formulation of the problem, and field solutions at zeroth through third order are considered in Sections 3–6. Reflected and transmitted powers are discussed in Section 7, and some example results illustrate higher-order SPM contributions in Section 8. Conclusions are presented in Section 9.

2. FORMULATION

Consider a deterministic periodic surface profile, $z = f(x, y)$, with periods P_x and P_y in the x and y directions respectively, that separates free space (permittivity ϵ_0, permeability μ_0) for $z > f(x, y)$ from a homogeneous nonmagnetic dielectric medium with permittivity $\epsilon_d = \epsilon\epsilon_0$ for $z < f(x, y)$. This periodic surface can also be expressed in terms of its Fourier series coefficients,

$$f(x, y) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \exp\left(i\frac{2\pi n x}{P_x}\right)\exp\left(i\frac{2\pi m y}{P_y}\right) h_{n,m}, \tag{1}$$

$$h_{n,m} = \frac{1}{P_x}\frac{1}{P_y}\int_0^{P_x}\mathrm{d}x\int_0^{P_y}\mathrm{d}y \exp\left(-i\frac{2\pi n x}{P_x}\right) \times \exp\left(-i\frac{2\pi m y}{P_y}\right) f(x, y). \tag{2}$$

Henceforth it will be assumed that all sums are from $-\infty$ to ∞ unless otherwise notated. With this description the surface derivatives are

$$\frac{\partial f}{\partial x} = \sum_n \sum_m \exp\left(i\frac{2\pi n x}{P_x}\right)\exp\left(i\frac{2\pi m y}{P_y}\right) i\frac{2\pi n}{P_x} h_{n,m}, \tag{3}$$

$$\frac{\partial f}{\partial y} = \sum_n \sum_m \exp\left(i\frac{2\pi n x}{P_x}\right)\exp\left(i\frac{2\pi m y}{P_y}\right) i\frac{2\pi m}{P_y} h_{n,m}. \tag{4}$$

Denoting the operator in Eq. (2) as $\mathcal{F}$, that is,

$$\mathcal{F}\{f(x, y)\}(n, m) = \frac{1}{P_x}\frac{1}{P_y}\int_0^{P_x}\mathrm{d}x\int_0^{P_y}\mathrm{d}y \exp\left(-i\frac{2\pi n x}{P_x}\right) \times \exp\left(-i\frac{2\pi m y}{P_y}\right) f(x, y), \tag{5}$$

it can be shown for N an integer greater than 1 that

$$\mathcal{F}\{f(x, y)^N\}(n, m) = \left(\sum_{n_1}\sum_{m_1}\right)\left(\sum_{n_2}\sum_{m_2}\right)(\cdots)\left(\sum_{n_{N-1}}\sum_{m_{N-1}}\right) h_{n_1,m_1}h_{n_2,m_2}\cdots h_{n_{N-1},m_{N-1}} h_{n-n_1-n_2-\cdots-n_{N-1},\, m-m_1-m_2-\cdots-m_{N-1}} \tag{6}$$

$$\mathcal{F}\left\{\frac{\partial f}{\partial x} f(x, y)^{N-1}\right\}(n, m) = \left(\sum_{n_1}\sum_{m_1}\right)\left(\sum_{n_2}\sum_{m_2}\right)(\cdots)\left(\sum_{n_{N-1}}\sum_{m_{N-1}}\right) h_{n_1,m_1}h_{n_2,m_2}\cdots h_{n_{N-1},m_{N-1}} h_{n-n_1-n_2-\cdots-n_{N-1},\, m-m_1-m_2-\cdots-m_{N-1}} \times \left[i\frac{2\pi}{P_x}(n - n_1 - n_2 - \cdots - n_{N-1})\right] \tag{7}$$

$$\mathcal{F}\left\{\frac{\partial f}{\partial y} f(x, y)^{N-1}\right\}(n, m) = \left(\sum_{n_1}\sum_{m_1}\right)\left(\sum_{n_2}\sum_{m_2}\right)(\cdots)\left(\sum_{n_{N-1}}\sum_{m_{N-1}}\right) h_{n_1,m_1}h_{n_2,m_2}\cdots h_{n_{N-1},m_{N-1}} h_{n-n_1-n_2-\cdots-n_{N-1},\, m-m_1-m_2-\cdots-m_{N-1}} \times \left[i\frac{2\pi}{P_y}(m - m_1 - m_2 - \cdots - m_{N-1})\right]. \tag{8}$$

These relationships will be useful in considering higher-order terms in the SPM equations.

Consider an incident electromagnetic plane wave that illuminates this periodic surface from the free-space region, with electric and magnetic fields given by

$$\mathbf{E}^i = \hat{e}_i \exp(i\mathbf{k}_i \cdot r), \tag{9}$$

$$\mathbf{H}^i = \frac{\hat{k}_i \times \hat{e}_i}{\eta_0}\exp(i\mathbf{k}_i \cdot \mathbf{r}), \tag{10}$$

where $\hat{e}_i$ represents the polarization vector of the incident electric field,

$$\mathbf{k}_i = k_0\hat{k}_i = \hat{x}k_{xi} + \hat{y}k_{yi} - \hat{z}k_{zi} \tag{11}$$

represents the propagation vector of the incident plane wave with wave number $k_0 = 2\pi/\lambda$,

$$\mathbf{r} = \hat{x}x + \hat{y}y + \hat{z}z \tag{12}$$

is a position vector in Cartesian space, and $\eta_0 = \sqrt{\mu_0/\epsilon_0}$ is the impedance of free space. Note that an $\exp(-i\omega t)$ time convention is assumed.

Under the Rayleigh hypothesis, the scattered field consists of a sum of upgoing plane waves (or Floquet modes), which can be written as

$$\mathbf{E}^s = \sum_m \sum_n [\hat{h}_s^{n,m}\alpha_{n,m} + \hat{v}_s^{n,m}\beta_{n,m}]\exp(i\mathbf{k}_s^{n,m} \cdot \mathbf{r}), \tag{13}$$

$$\mathbf{H}^s = \frac{1}{\eta_0}\sum_m \sum_n [-\hat{v}_s^{n,m}\alpha_{n,m} + \hat{h}_s^{n,m}\beta_{n,m}]\exp(i\mathbf{k}_s^{n,m} \cdot \mathbf{r}), \tag{14}$$

while transmitted fields consist only of downgoing plane waves, which can be written as

$$\mathbf{E}^t = \sum_m \sum_n [\hat{h}_s^{n,m}\gamma_{n,m} + \hat{v}_t^{n,m}\delta_{n,m}]\exp(i\mathbf{k}_t^{n,m} \cdot \mathbf{r}), \quad (15)$$

$$\mathbf{H}^t = \frac{1}{\eta_1}\sum_m \sum_n [-\hat{v}_t^{n,m}\gamma_{n,m} + \hat{h}_s^{n,m}\delta_{n,m}]\exp(i\mathbf{k}_t^{n,m} \cdot \mathbf{r}), \quad (16)$$

where $\eta_1 = \eta_0/\sqrt{\epsilon}$ is the impedance of the lower medium and α, β, γ, and δ are the unknown complex amplitudes of the scattered and transmitted Floquet modes. Scattered and transmitted plane-wave propagation vectors are defined by the Floquet theorem as

$$\mathbf{k}_s^{n,m} = \hat{x}k_{xn} + \hat{y}k_{ym} + \hat{z}k_{znm}, \quad (17)$$

$$\mathbf{k}_t^{n,m} = \hat{x}k_{xn} + \hat{y}k_{ym} - \hat{z}k_{z1nm}, \quad (18)$$

where

$$k_{xn} = k_{xi} + \frac{2\pi n}{P_x}, \quad (19)$$

$$k_{ym} = k_{yi} + \frac{2\pi m}{P_y}, \quad (20)$$

$$k_{\rho nm} = \sqrt{k_{xn}^2 + k_{ym}^2}, \quad (21)$$

$$k_{znm} = \sqrt{k_0^2 - k_{\rho nm}^2}, \quad (22)$$

$$k_{z1nm} = \sqrt{k_0^2\epsilon - k_{\rho nm}^2}. \quad (23)$$

Modes for which $k_{\rho nm}$ becomes greater than k_0 or k_1 have k_{znm} and k_{z1nm}, respectively, defined so that attenuation occurs as fields propagate away from the surface boundary. Orthogonal horizontal and vertical polarization vectors for the incident, scattered, and transmitted fields are defined as

$$\hat{h}_i = \hat{x}\frac{k_{yi}}{k_{\rho i}} - \hat{y}\frac{k_{xi}}{k_{\rho i}}, \quad (24)$$

$$\hat{h}_s^{n,m} = \hat{x}\frac{k_{ym}}{k_{\rho nm}} - \hat{y}\frac{k_{xn}}{k_{\rho nm}}, \quad (25)$$

$$\hat{h}_t^{n,m} = \hat{h}_s^{n,m}, \quad (26)$$

$$\hat{v}_i = \hat{x}\frac{k_{xi}k_{zi}}{k_0k_{\rho i}} + \hat{y}\frac{k_{yi}k_{zi}}{k_0k_{\rho i}} + \hat{z}\frac{k_{\rho i}}{k_0}, \quad (27)$$

$$\hat{v}_s^{n,m} = -\hat{x}\frac{k_{xn}k_{znm}}{k_0k_{\rho nm}} - \hat{y}\frac{k_{ym}k_{znm}}{k_0k_{\rho nm}} + \hat{z}\frac{k_{\rho nm}}{k_0}, \quad (28)$$

$$\hat{v}_t^{n,m} = \hat{x}\frac{k_{xn}k_{z1nm}}{k_1k_{\rho nm}} + \hat{y}\frac{k_{ym}k_{z1nm}}{k_1k_{\rho nm}} + \hat{z}\frac{k_{\rho nm}}{k_1}, \quad (29)$$

where $k_1 = k_0\sqrt{\epsilon}$ is the wave number in the lower medium.

Boundary conditions on the interface specify that tangential electric and magnetic fields must be continuous:

$$(\hat{z} - \partial\mathbf{f}) \times (\mathbf{E}^i + \mathbf{E}^s) = (\hat{z} - \partial\mathbf{f}) \times \mathbf{E}^t, \quad (30)$$

$$(\hat{z} - \partial\mathbf{f}) \times (\mathbf{H}^i + \mathbf{H}^s) = (\hat{z} - \partial\mathbf{f}) \times \mathbf{H}^t, \quad (31)$$

since a vector normal to the surface can be written as $\hat{z} - \partial\mathbf{f}$, where $\partial\mathbf{f} = \hat{x}(\partial f/\partial x) + \hat{y}(\partial f/\partial y)$. These equations can be rewritten as

$$\hat{z} \times \mathbf{E}^s - \hat{z} \times \mathbf{E}^t = -\hat{z} \times \mathbf{E}^i + \partial\mathbf{f} \times \mathbf{E}^i + \partial\mathbf{f} \times \mathbf{E}^s - \partial\mathbf{f} \times \mathbf{E}^t, \quad (32)$$

$$\hat{z} \times \mathbf{H}^s - \hat{z} \times \mathbf{H}^t = -\hat{z} \times \mathbf{H}^i + \partial\mathbf{f} \times \mathbf{H}^i + \partial\mathbf{f} \times \mathbf{H}^s - \partial\mathbf{f} \times \mathbf{H}^t. \quad (33)$$

Equations (32) and (33) are evaluated on the surface boundary, where $z = f(x, y)$. Substituting in the Rayleigh hypothesis fields (13)–(16) and considering only the x and y components of the above equations, two two-component vector equations are obtained:

$$\begin{aligned}\sum_m \sum_n \exp\left(i\frac{2\pi nx}{P_x}\right)\exp\left(i\frac{2\pi my}{P_y}\right)\Big\{ (\hat{z} \times \hat{h}_s^{n,m})[\alpha_{n,m}\exp(ik_{znm}z) - \gamma_{n,m}\exp(-ik_{z1nm}z)] \\ + (\hat{z} \times \hat{v}_s^{n,m})\left[\beta_{n,m}\exp(ik_{znm}z) + \frac{k_0k_{z1nm}}{k_1k_{znm}}\delta_{n,m}\exp(-ik_{z1nm}z)\right]\Big\} \\ = -(\hat{z} \times \hat{e}_i)\exp(-ik_{zi}z) + (\partial\mathbf{f} \times \hat{e}_i)\exp(-ik_{zi}z) + \sum_m \sum_n \exp\left(i\frac{2\pi nx}{P_x}\right)\exp\left(i\frac{2\pi my}{P_y}\right) \\ \times \left\{(\partial\mathbf{f} \times \hat{v}_s^{n,m})\left[\beta_{n,m}\exp(ik_{znm}z) - \frac{k_0}{k_1}\delta_{n,m}\exp(-ik_{z1nm}z)\right]\right\},\end{aligned} \quad (34)$$

$$\sum_m \sum_n \exp\left(i\frac{2\pi nx}{P_x}\right)\exp\left(i\frac{2\pi my}{P_y}\right)\left\{(-\hat{z}\times\hat{v}_s^{n,m})\left[\alpha_{n,m}\exp(ik_{znm}z)+\frac{k_{z1nm}}{k_{znm}}\gamma_{n,m}\exp(-ik_{z1nm}z)\right]\right.$$
$$\left.+(\hat{z}\times\hat{h}_s^{n,m})\left[\beta_{n,m}\exp(ik_{znm}z)-\frac{k_1}{k_0}\delta_{n,m}\exp(-ik_{z1nm}z)\right]\right\}$$
$$=-(\hat{z}\times\hat{k}_i\times\hat{e}_i)\exp(-ik_{zi}z)+(\partial\mathbf{f}\times\hat{k}_i\times\hat{e}_i)\exp(-ik_{zi}z)+\sum_m\sum_n\exp\left(i\frac{2\pi nx}{P_x}\right)\exp\left(i\frac{2\pi my}{P_y}\right)$$
$$\times\{(-\partial\mathbf{f}\times\hat{v}_s^{n,m})[\alpha_{n,m}\exp(ik_{znm}z)-\gamma_{n,m}\exp(-ik_{z1nm}z)]\}. \tag{35}$$

At this point, a small height expansion is used by expanding the exponentials in Eqs. (34) and (35) in power series:

$$\exp(\pm ik_z z)=\sum_{q=0}^{\infty}\frac{(\pm ik_z z)^q}{q!} \tag{36}$$

and by substituting a perturbation series for the unknowns

$$\alpha_{n,m}=\alpha_{n,m}^{(0)}+\alpha_{n,m}^{(1)}+\cdots=\sum_{l=0}^{\infty}\alpha_{n,m}^{(l)} \tag{37}$$

with similar definitions for β, γ, and δ. Perturbation series terms are defined so that the lth term is of order f^l or equivalent combinations of f and its derivatives, since $\partial f/\partial x$ and $\partial f/\partial y$ are assumed to be the same order as f.

Substituting these expansions and collecting terms of order z^N (or f^N), Eqs. (34) and (35) simplify to

$$\sum_m\sum_n\exp\left(i\frac{2\pi nx}{P_x}\right)\exp\left(i\frac{2\pi my}{P_y}\right)\left[(\hat{z}\times\hat{h}_s^{n,m})(\alpha_{n,m}^{(N)}-\gamma_{n,m}^{(N)})+(\hat{z}\times\hat{v}_s^{n,m})\left(\beta_{n,m}^{(N)}+\frac{k_0k_{z1nm}}{k_1k_{znm}}\delta_{n,m}^{(N)}\right)\right]$$
$$=-(\hat{z}\times\hat{e}_i)\frac{(-ik_{zi}z)^N}{N!}+(\partial\mathbf{f}\times\hat{e}_i)\frac{(-ik_{zi}z)^{N-1}}{(N-1)!}-\left(\sum_{l=0}^{N-1}\frac{(iz)^{N-l}}{(N-l)!}\sum_m\sum_n\exp\left(i\frac{2\pi nx}{P_x}\right)\exp\left(i\frac{2\pi my}{P_y}\right)\right.$$
$$\left.\times\left\{(\hat{z}\times\hat{h}_s^{n,m})[\alpha_{n,m}^{(l)}(k_{znm})^{N-l}-\gamma_{n,m}^{(l)}(-k_{z1nm})^{N-l}]+(\hat{z}\times\hat{v}_s^{n,m})\left[\beta_{n,m}^{(l)}(k_{znm})^{N-l}+\frac{k_0k_{z1nm}}{k_1k_{znm}}\delta_{n,m}^{(l)}(-k_{z1nm})^{N-l}\right]\right\}\right)$$
$$+\sum_{l=0}^{N-1}\frac{(iz)^{N-l-1}}{(N-l-1)!}\sum_m\sum_n\exp\left(i\frac{2\pi nx}{P_x}\right)\exp\left(i\frac{2\pi my}{P_y}\right)\left\{(\partial\mathbf{f}\times\hat{v}_s^{n,m})\left[\beta_{n,m}^{(l)}(k_{znm})^{N-l-1}\right.\right.$$
$$\left.\left.-\frac{k_0}{k_1}\delta_{n,m}^{(l)}(-k_{z1nm})^{N-l-1}\right]\right\}, \tag{38}$$

$$\sum_m\sum_n\exp\left(i\frac{2\pi nx}{P_x}\right)\exp\left(i\frac{2\pi my}{P_y}\right)\left[(-\hat{z}\times\hat{v}_s^{n,m})\left(\alpha_{n,m}^{(N)}+\frac{k_{z1nm}}{k_{znm}}\gamma_{n,m}^{(N)}\right)+(\hat{z}\times\hat{h}_s^{n,m})\left(\beta_{n,m}^{(N)}-\frac{k_1}{k_0}\delta_{n,m}^{(N)}\right)\right]$$
$$=-(\hat{z}\times\hat{k}_i\times\hat{e}_i)\frac{(-ik_{zi}z)^N}{N!}+(\partial\mathbf{f}\times\hat{k}_i\times\hat{e}_i)\frac{(-ik_{zi}z)^{N-1}}{(N-1)!}-\left(\sum_{l=0}^{N-1}\frac{(iz)^{N-l}}{(N-l)!}\sum_m\sum_n\exp\left(i\frac{2\pi nx}{P_x}\right)\exp\left(i\frac{2\pi my}{P_y}\right)\right.$$
$$\left.\times\left\{-(\hat{z}\times\hat{v}_s^{n,m})\left[\alpha_{n,m}^{(l)}(k_{znm})^{N-l}+\frac{k_{z1nm}}{k_{znm}}\gamma_{n,m}^{(l)}(-k_{z1nm})^{N-l}\right]+(\hat{z}\times\hat{h}_s^{n,m})\left[\beta_{n,m}^{(l)}(k_{znm})^{N-l}-\frac{k_1}{k_0}\delta_{n,m}^{(l)}(-k_{z1nm})^{N-l}\right]\right\}\right)$$
$$+\sum_{l=0}^{N-l}\frac{(iz)^{N-l-1}}{(N-l-1)!}\sum_m\sum_n\exp\left(i\frac{2\pi nx}{P_x}\right)\exp\left(i\frac{2\pi my}{P_y}\right)\{(-\partial\mathbf{f}\times\hat{v}_s^{n,m})[\alpha_{n,m}^{(l)}(k_{znm})^{N-l-1}-\gamma_{n,m}^{(l)}(-k_{z1nm})^{N-l-1}]\}. \tag{39}$$

Note that when $N = 0$, terms involving $N - 1$ are not included.

These equations hold in the space (i.e., x–y) domain. Defining the right-hand sides of Eqs. (38) and (39) (which by definition have only x and y components) as $\mathbf{S}_E^{(N)}(x, y)$ and $\mathbf{S}_H^{(N)}(x, y)$, respectively, the equations are

$$\sum_m \sum_n \exp\left(i\frac{2\pi n x}{P_x}\right)\exp\left(i\frac{2\pi m y}{P_y}\right) \times \left[(\hat{z} \times \hat{h}_s^{n,m})(\alpha_{n,m}^{(N)} - \gamma_{n,m}^{(N)}) + (\hat{z} \times \hat{v}_s^{n,m}) \times \left(\beta_{n,m}^{(N)} + \frac{k_0 k_{z1nm}}{k_1 k_{znm}}\delta_{n,m}^{(N)}\right)\right] = \mathbf{S}_E^{(N)}(x, y), \quad (40)$$

$$\sum_m \sum_n \exp\left(i\frac{2\pi n x}{P_x}\right)\exp\left(i\frac{2\pi m y}{P_y}\right) \times \left[(-\hat{z} \times \hat{v}_s^{n,m})\left(\alpha_{n,m}^{(N)} + \frac{k_{z1nm}}{k_{znm}}\gamma_{n,m}^{(N)}\right) + (\hat{z} \times \hat{h}_s^{n,m}) \times \left(\beta_{n,m}^{(N)} - \frac{k_1}{k_0}\delta_{n,m}^{(N)}\right)\right] = \mathbf{S}_H^{(N)}(x, y). \quad (41)$$

Applying the $\mathcal{F}$ operator to both sides of this equation and using the orthogonality properties of complex exponentials yields

$$(\hat{z} \times \hat{h}_s^{n',m'})(\alpha_{n',m'}^{(N)} - \gamma_{n',m'}^{(N)}) + (\hat{z} \times \hat{v}_s^{n',m'}) \times \left(\beta_{n',m'}^{(N)} + \frac{k_0 k_{z1n'm'}}{k_1 k_{zn'm'}}\delta_{n',m'}^{(N)}\right) = \mathcal{F}\{\mathbf{S}_E^{(N)}(x, y)\}(n', m'), \quad (42)$$

$$(-\hat{z} \times \hat{v}_s^{n',m'})\left(\alpha_{n',m'}^{(N)} + \frac{k_{z1n'm'}}{k_{zn'm'}}\gamma_{n',m'}^{(N)}\right) + (\hat{z} \times \hat{h}_s^{n',m'}) \times \left(\beta_{n',m'}^{(N)} - \frac{k_1}{k_0}\delta_{n',m'}^{(N)}\right) = \mathcal{F}\{\mathbf{S}_H^{(N)}(x, y)\}(n', m'), \quad (43)$$

which can be solved to determine

$$\alpha_{n',m'}^{(N)} = \left(\frac{-k_0}{k_{zn'm'} + k_{z1n'm'}}\right)\left(\hat{v}_s^{n',m'} \cdot \mathcal{F}\{\mathbf{S}_E^{(N)}(x, y)\} \times \frac{k_{z1n'm'}}{k_{zn'm'}} + \hat{h}_s^{n',m'} \cdot \mathcal{F}\{\mathbf{S}_H^{(N)}(x, y)\}\right), \quad (44)$$

$$\beta_{n',m'}^{(N)} = \left(\frac{k_0}{\epsilon k_{zn'm'} + k_{z1n'm'}}\right)\left(\epsilon\hat{h}_s^{n',m'} \cdot \mathcal{F}\{\mathbf{S}_E^{(N)}(x, y)\} - \hat{v}_s^{n',m'} \cdot \mathcal{F}\{\mathbf{S}_H^{(N)}(x, y)\}\frac{k_{z1n'm'}}{k_{zn'm'}}\right), \quad (45)$$

$$\gamma_{n',m'}^{(N)} = \left(\frac{k_0}{k_{zn'm'} + k_{z1n'm'}}\right)(\hat{v}_s^{n',m'} \cdot \mathcal{F}\{\mathbf{S}_E^{(N)}(x, y)\} - \hat{h}_s^{n',m'} \cdot \mathcal{F}\{\mathbf{S}_H^{(N)}(x, y)\}), \quad (46)$$

$$\delta_{n',m'}^{(N)} = \left(\frac{k_1}{\epsilon k_{zn'm'} + k_{z1n'm'}}\right)(\hat{h}_s^{n',m'} \cdot \mathcal{F}\{\mathbf{S}_E^{(N)}(x, y)\} + \hat{v}_s^{n',m'} \cdot \mathcal{F}\{\mathbf{S}_H^{(N)}(x, y)\}), \quad (47)$$

where the (n', m') index of the $\mathcal{F}$ operators has been dropped for convenience. Since $\mathbf{S}_E^{(N)}$ and $\mathbf{S}_H^{(N)}$ depend only on solutions of order less than N and on the known incident field, the above equations represent the unknown field amplitudes at order N in terms of known quantities. These equations can be easily implemented on a computer through use of the fast Fourier transform for the $\mathcal{F}$ operator, permitting solution of the SPM equations to arbitrary order. However, only analytical solutions are considered in this paper.

3. ZEROTH-ORDER SOLUTION

The solution at zeroth order is particularly simple since the sums over l in the definition of $\mathbf{S}_E^{(N)}$ and $\mathbf{S}_H^{(N)}$ vanish, leaving only

$$\mathbf{S}_E^{(0)} = -\hat{z} \times \hat{e}_i, \quad (48)$$

$$\mathbf{S}_H^{(0)} = -\hat{z} \times \hat{k}_i \times \hat{e}_i. \quad (49)$$

Fourier series coefficients (i.e., after applying the $\mathcal{F}$ operator) for these functions are simply Kronecker delta functions, indicating that the field amplitudes $\alpha_{n',m'}^{(0)}$ through $\delta_{n',m'}^{(0)}$ are nonzero only for $n' = m' = 0$, the specularly reflected and transmitted plane waves. Field amplitude results are

$$\alpha_{0,0}^{(0)} = \frac{k_{zi} - k_{z1i}}{k_{zi} + k_{z1i}} = \Gamma_H, \quad (50)$$

$$\beta_{0,0}^{(0)} = 0, \quad (51)$$

$$\gamma_{0,0}^{(0)} = 1 + \Gamma_H, \quad (52)$$

$$\delta_{0,0}^{(0)} = 0, \quad (53)$$

for a horizontally polarized incident field (i.e., $\hat{e}_i = \hat{h}_i$) and

$$\alpha_{0,0}^{(0)} = 0, \quad (54)$$

$$\beta_{0,0}^{(0)} = \frac{\epsilon k_{zi} - k_{z1i}}{\epsilon k_{zi} + k_{z1i}} = \Gamma_V, \quad (55)$$

$$\gamma_{0,0}^{(0)} = 0, \quad (56)$$

$$\delta_{0,0}^{(0)} = \frac{k_0}{k_1}(1 + \Gamma_V), \quad (57)$$

for a vertically polarized incident field.

To simplify the solution at higher than zeroth order, these zeroth order field solutions can be combined with the incident fields to produce a new $l = 0$ term in $\mathbf{S}_E^{(N)}$ and $\mathbf{S}_H^{(N)}$. Contributions from these new terms to field coefficients at order N can then be computed following Eqs. (44)–(47) and are found to be

$$\alpha_{n',m'}^{(N,0)} = \frac{i^N}{N!}\left(\frac{-k_0}{k_{zn'm'} + k_{z1n'm'}}\right)\left[-\mathcal{F}\{z^N\}c_{i,n'}\left(-\frac{k_{z1n'm'}}{k_0}R_{1h} - \frac{k_{zi}}{k_0}R_{2h}\right) + \left(\frac{k_{ym'}}{k_{\rho n'm'}}\mathcal{F}\left\{\frac{\partial f}{\partial y}z^{N-1}\right\} + \frac{k_{xn'}}{k_{\rho n'm'}}\mathcal{F}\left\{\frac{\partial f}{\partial x}z^{N-1}\right\}\right)(-iN)R_{3h}\right], \tag{58}$$

$$\beta_{n',m'}^{(N,0)} = \frac{i^N}{N!}\left(\frac{k_0}{\epsilon k_{zn'm'} + k_{z1n'm'}}\right)\left[-\mathcal{F}\{z^N\}s_{i,n'}\left(\epsilon R_{1h} + \frac{k_{zi}k_{z1n'm'}}{k_0^2}R_{2h}\right) - \frac{k_{z1n'm'}}{k_0}\left(\frac{k_{ym'}}{k_{\rho n'm'}}\mathcal{F}\left\{\frac{\partial f}{\partial x}z^{N-1}\right\} - \frac{k_{xn'}}{k_{\rho n'm'}}\mathcal{F}\left\{\frac{\partial f}{\partial y}z^{N-1}\right\}\right)(-iN)R_{3h}\right], \tag{59}$$

$$\gamma_{n',m'}^{(N,0)} = \frac{i^N}{N!}\left(\frac{k_0}{k_{zn'm'} + k_{z1n'm'}}\right)\left[-\mathcal{F}\{z^N\}c_{i,n'}\left(-\frac{k_{zn'm'}}{k_0}R_{1h} + \frac{k_{zi}}{k_0}R_{2h}\right) - \left(\frac{k_{ym'}}{k_{\rho n'm'}}\mathcal{F}\left\{\frac{\partial f}{\partial y}z^{N-1}\right\} + \frac{k_{xn'}}{k_{\rho n'm'}}\mathcal{F}\left\{\frac{\partial f}{\partial x}z^{N-1}\right\}\right)(-iN)R_{3h}\right], \tag{60}$$

$$\delta_{n',m'}^{(N,0)} = \frac{i^N}{N!}\left(\frac{k_1}{\epsilon k_{zn'm'} + k_{z1n'm'}}\right)\left[-\mathcal{F}\{z^N\}s_{i,n'}\left(R_{1h} - \frac{k_{zi}k_{zn'm'}}{k_0^2}R_{2h}\right) + \frac{k_{zn'm'}}{k_0}\left(\frac{k_{ym'}}{k_{\rho n'm'}}\mathcal{F}\left\{\frac{\partial f}{\partial x}z^{N-1}\right\} - \frac{k_{xn'}}{k_{\rho n'm'}}\mathcal{F}\left\{\frac{\partial f}{\partial y}z^{N-1}\right\}\right)(-iN)R_{3h}\right], \tag{61}$$

for a horizontally polarized incident field, where

$$R_{1h} = \Gamma_H(k_{zi})^N + (-k_{zi})^N - (1 + \Gamma_H)(-k_{z1i})^N, \tag{62}$$

$$R_{2h} = \Gamma_H(k_{zi})^N - (-k_{zi})^N + \frac{k_{z1i}}{k_{zi}}(1 + \Gamma_H)(-k_{z1i})^N, \tag{63}$$

$$R_{3h} = -\frac{k_{\rho i}}{k_0}((-k_{zi})^{N-1} + \Gamma_H(k_{zi})^{N-1} - (1 + \Gamma_H)(-k_{z1i})^{N-1}), \tag{64}$$

and the $(N, 0)$ notation refers to the fact that the above quantities are the new $l = 0$ term contributions to the field amplitudes at order N. Again in Eqs. (58)–(61) the indices (n', m') have been dropped after the $\mathcal{F}$ operators for convenience, and a new notation for sine and cosine functions has been introduced:

$$c_{i,n'} = \frac{k_{xi}k_{xn'} + k_{yi}k_{ym'}}{k_{\rho i}k_{\rho n'm'}}, \tag{65}$$

$$s_{i,n'} = \frac{k_{xi}k_{ym'} - k_{yi}k_{xn'}}{k_{\rho i}k_{\rho n'm'}}. \tag{66}$$

This notation will be generalized and used throughout the paper: The first subscript in the c_{n_1,n_2} or s_{n_1,n_2} functions refers to the subscript to be applied to the first $k_{x_{n_1}}$ quantity, and the second subscript refers to the subscript type to be applied to the quantity multiplying the first $k_{x_{n_1}}$. Note that the k_ρ terms in the denominator also contain the appropriate subscripts.

For a vertically polarized incident field,

$$\alpha_{n',m'}^{(N,0)} = \frac{i^N}{N!}\left(\frac{-k_0}{k_{zn'm'} + k_{z1n'm'}}\right)\left[-\mathcal{F}\{z^N\}s_{i,n'}\left(\frac{k_{zi}k_{z1n'm'}}{k_0^2}R_{1v} + R_{2v}\right) + \frac{k_{z1n'm'}}{k_0}\left(\frac{k_{ym'}}{k_{\rho n'm'}}\mathcal{F}\left\{\frac{\partial f}{\partial x}z^{N-1}\right\} - \frac{k_{xn'}}{k_{\rho n'm'}}\mathcal{F}\left\{\frac{\partial f}{\partial y}z^{N-1}\right\}\right)(-iN)R_{3v}\right], \tag{67}$$

$$\beta_{n',m'}^{(N,0)} = \frac{i^N}{N!}\left(\frac{k_0}{\epsilon k_{zn'm'} + k_{z1n'm'}}\right)\Bigg[-\mathcal{F}\{z^N\}c_{i,n'}\left(\frac{\epsilon k_{zi}}{k_0}R_{1v} + \frac{k_{z1n'm'}}{k_0}R_{2v}\right) + \epsilon\left(\frac{k_{ym'}}{k_{\rho n'm'}}\mathcal{F}\left\{\frac{\partial f}{\partial y}z^{N-1}\right\} + \frac{k_{xn'}}{k_{\rho n'm'}}\mathcal{F}\left\{\frac{\partial f}{\partial x}z^{N-1}\right\}\right)(-iN)R_{3v}\Bigg], \quad (68)$$

$$\gamma_{n',m'}^{(N,0)} = \frac{i^N}{N!}\left(\frac{k_0}{k_{zn'm'} + k_{z1n'm'}}\right)\Bigg[-\mathcal{F}\{z^N\}s_{i,n'}\left(\frac{k_{zi}k_{zn'm'}}{k_0^2}R_{1v} - R_{2v}\right) + \frac{k_{zn'm'}}{k_0}\left(\frac{k_{ym'}}{k_{\rho n'm'}}\mathcal{F}\left\{\frac{\partial f}{\partial x}z^{N-1}\right\} - \frac{k_{xn'}}{k_{\rho n'm'}}\mathcal{F}\left\{\frac{\partial f}{\partial y}z^{N-1}\right\}\right)(-iN)R_{3v}\Bigg], \quad (69)$$

$$\delta_{n',m'}^{(N,0)} = \frac{i^N}{N!}\left(\frac{k_1}{\epsilon k_{zn'm'} + k_{z1n'm'}}\right)\Bigg[-\mathcal{F}\{z^N\}c_{i,n'}\left(\frac{k_{zi}}{k_0}R_{1v} - \frac{k_{zn'm'}}{k_0}R_{2v}\right) + \left(\frac{k_{ym'}}{k_{\rho n'm'}}\mathcal{F}\left\{\frac{\partial f}{\partial y}z^{N-1}\right\} + \frac{k_{xn'}}{k_{\rho n'm'}}\mathcal{F}\left\{\frac{\partial f}{\partial x}z^{N-1}\right\}\right)(-iN)R_{3v}\Bigg], \quad (70)$$

where

$$R_{1v} = \Gamma_V(k_{zi})^N - (-k_{zi})^N + \frac{k_{z1i}}{\epsilon k_{zi}}(1 + \Gamma_V)(-k_{z1i})^N, \quad (71)$$

$$R_{2v} = \Gamma_V(k_{zi})^N + (-k_{zi})^N - (1 + \Gamma_V)(-k_{z1i})^N, \quad (72)$$

$$R_{3v} = \frac{k_{\rho i}}{k_0}\left[(-k_{zi})^{N-1} + \Gamma_V(k_{zi})^{N-1} - \frac{1 + \Gamma_V}{\epsilon}(-k_{z1i})^{N-1}\right]. \quad (73)$$

Equations (6)–(8) with indices n' and m' can be applied to simplify the $\mathcal{F}$ operators once the order N is specified.

4. FIRST-ORDER SOLUTION

Substituting $N = 1$ into Eqs. (44)–(47) shows that knowledge of $\mathbf{S}_E^{(1)}$ and $\mathbf{S}_H^{(1)}$ is required. Examination of Eqs. (38)–(39) reveals that the $\alpha_{n',m'}^{(N,0)}$ through $\delta_{n',m'}^{(N,0)}$ terms described above are sufficient to determine field amplitudes at first order, and all terms are found to be directly proportional to $h_{n',m'}$. A general form for first-order solutions is

$$\zeta_{n',m'}^{(1)} = h_{n',m'}g_\zeta^{(1)}(k_{xn'}, k_{ym'}), \quad (74)$$

where $\zeta = \alpha$, β, γ, or δ and $g_\zeta^{(1)}$ is a corresponding function. Solutions for a horizontally polarized incident field are

$$g_\alpha^{(1)} = \frac{-2ik_{zi}(k_0^2 - k_1^2)}{(k_{zn'm'} + k_{z1n'm'})(k_{zi} + k_{z1i})}c_{i,n'}, \quad (75)$$

$$g_\beta^{(1)} = \frac{-2ik_{zi}(k_0^2 - k_1^2)}{(\epsilon k_{zn'm'} + k_{z1n'm'})(k_{zi} + k_{z1i})}\left(\frac{k_{z1n'm'}}{k_0}\right)s_{i,n'}, \quad (76)$$

$$g_\gamma^{(1)} = g_\alpha^{(1)}, \quad (77)$$

$$g_\delta^{(1)} = -g_\beta^{(1)}\left(\frac{k_{zn'm'}k_1}{k_{z1n'm'}k_0}\right), \quad (78)$$

and

$$g_\alpha^{(1)} = \frac{-2ik_{zi}(k_0^2 - k_1^2)}{(k_{zn'm'} + k_{z1n'm'})(\epsilon k_{zi} + k_{z1i})}\left(\frac{k_{z1i}}{k_0}\right)s_{i,n'}, \quad (79)$$

$$g_\beta^{(1)} = \frac{-2ik_{zi}(k_0^2 - k_1^2)}{(\epsilon k_{zn'm'} + k_{z1n'm'})(\epsilon k_{zi} + k_{z1i})} \times \left(\frac{\epsilon k_{\rho i}k_{\rho n'm'}}{k_0^2} - \frac{k_{z1i}k_{z1n'm'}}{k_0^2}c_{i,n'}\right), \quad (80)$$

$$g_\gamma^{(1)} = g_\alpha^{(1)}, \quad (81)$$

$$g_\delta^{(1)} = \frac{-2ik_{zi}(k_0^2 - k_1^2)\sqrt{\epsilon}}{(\epsilon k_{zn'm'} + k_{z1n'm'})(\epsilon k_{zi} + k_{z1i})} \times \left(\frac{k_{\rho i}k_{\rho n'm'}}{k_0^2} + \frac{k_{z1i}k_{zn'm'}}{k_0^2}c_{i,n'}\right), \quad (82)$$

for a vertically polarized incident field. These results illustrate the "Bragg scatter" phenomenon of first-order perturbation theory, since scattered fields at a particular angle [i.e., (n', m')] are directly proportional to the amplitude of a particular surface Fourier component.

5. SECOND-ORDER SOLUTION

For $N = 2$, $\mathbf{S}_E^{(2)}$ and $\mathbf{S}_H^{(2)}$ are required and contain the $l = 0$ contributions described above and contributions from an $l = 1$ term. Recognizing that the $l > 0$ terms at any order N consist of the product of two functions of space [i.e., $z^{(N-l)}$ and the sums over m and n in Eqs. (38) and (39)], the convolution theorem can be applied to determine the $l > 0$ term contributions to $\alpha^{(N)}$ through $\delta^{(N)}$. The general equations that result are

$$\alpha^{(N,r)}_{n',m'} = \frac{-k_0}{k_{zn'm'} + k_{z1n'm'}}\Bigg(-\sum_{l=1}^{N-1} \frac{i^{N-l}}{(N-l)!}\sum_m \sum_n \mathcal{F}\{z^{N-l}\}(n'-n, m'-m)$$
$$\times \Bigg\{ c_{n,n'}\Bigg[-\alpha^{(l)}_{n,m}(k_{znm})^{N-l}\Bigg(\frac{k_{z1n'm'} + k_{znm}}{k_0}\Bigg) + \gamma^{(l)}_{n,m}(-k_{z1nm})^{N-l}\Bigg(\frac{k_{z1n'm'} - k_{z1nm}}{k_0}\Bigg)\Bigg]$$
$$+ s_{n,n'}\Bigg[\beta^{(l)}_{n,m}(k_{znm})^{N-l}\Bigg(\frac{k_0^2 + k_{z1n'm'}k_{znm}}{k_0^2}\Bigg) + \delta^{(l)}_{n,m}(-k_{z1nm})^{N-l}\Bigg(\frac{-k_1^2 + k_{z1n'm'}k_{z1nm}}{k_1 k_0}\Bigg)\Bigg]\Bigg\}$$
$$+ \sum_{l=1}^{N-1} \frac{i^{N-l-1}}{(N-l-1)!}\sum_m \sum_n \frac{k_{\rho nm}}{k_0}\Bigg\{\Bigg[\frac{k_{ym'}}{k_{\rho n'm'}}\mathcal{F}\Bigg\{\frac{\partial f}{\partial x}z^{N-l-1}\Bigg\}(n'-n, m'-m)$$
$$- \frac{k_{xn'}}{k_{\rho n'm'}}\mathcal{F}\Bigg\{\frac{\partial f}{\partial y}z^{N-l-1}\Bigg\}(n'-n, m'-m)\Bigg]\Bigg(\frac{k_{z1n'm'}}{k_0}\Bigg)\Bigg[\beta^{(l)}_{n,m}(k_{znm})^{N-l-1} - \frac{k_0}{k_1}\delta^{(l)}_{n,m}(-k_{z1nm})^{N-l-1}\Bigg]$$
$$- \Bigg[\frac{k_{ym'}}{k_{\rho n'm'}}\mathcal{F}\Bigg\{\frac{\partial f}{\partial y}z^{N-l-1}\Bigg\}(n'-n, m'-m) + \frac{k_{xn'}}{k_{\rho n'm'}}\mathcal{F}\Bigg\{\frac{\partial f}{\partial x}z^{N-l-1}\Bigg\}(n'-n, m'-m)\Bigg]$$
$$\times [\alpha^{(l)}_{n,m}(k_{znm})^{N-l-1} - \gamma^{(l)}_{n,m}(-k_{z1nm})^{N-l-1}]\Bigg\}\Bigg), \tag{83}$$

$$\beta^{(N,r)}_{n',m'} = \frac{k_0}{\epsilon k_{zn'm'} + k_{z1n'm'}}\Bigg(-\sum_{l=1}^{N-1} \frac{i^{N-1}}{(N-l)!}\sum_m \sum_n \mathcal{F}\{z^{N-1}\}(n'-n, m'-m)$$
$$\times \Bigg\{ s_{n,n'}\Bigg[\alpha^{(l)}_{n,m}(k_{znm})^{N-l}\Bigg(\frac{k_1^2 + k_{z1n'm'}k_{znm}}{k_0^2}\Bigg) + \gamma^{(l)}_{n,m}(-k_{z1nm})^{N-l}\Bigg(\frac{-k_1^2 + k_{z1n'm'}k_{z1nm}}{k_0^2}\Bigg)\Bigg]$$
$$+ c_{n,n'}\Bigg[\beta^{(l)}_{n,m}(k_{znm})^{N-l}\Bigg(\frac{\epsilon k_{znm} + k_{z1n'm'}}{k_0}\Bigg) + \delta^{(l)}_{n,m}(-k_{z1nm})^{N-l}\Bigg(\epsilon\frac{k_{z1nm} - k_{z1n'm'}}{k_1}\Bigg)\Bigg]\Bigg\}$$
$$+ \sum_{l=1}^{N-1} \frac{i^{N-l-1}}{(N-l-1)!}\sum_m \sum_n \frac{k_{\rho nm}}{k_0}\Bigg\{\Bigg[\frac{k_{ym'}}{k_{\rho n'm'}}\mathcal{F}\Bigg\{\frac{\partial f}{\partial x}z^{N-l-1}\Bigg\}(n'-n, m'-m)$$
$$- \frac{k_{xn'}}{k_{\rho n'm'}}\mathcal{F}\Bigg\{\frac{\partial f}{\partial y}z^{N-l-1}\Bigg\}(n'-n, m'-m)\Bigg]\frac{k_{z1n'm'}}{k_0}[\alpha^{(l)}_{n,m}(k_{znm})^{N-l-1} - \gamma^{(l)}_{n,m}(-k_{z1nm})^{N-l-1}]$$
$$+ \Bigg[\frac{k_{ym'}}{k_{\rho n'm'}}\mathcal{F}\Bigg\{\frac{\partial f}{\partial y}z^{N-l-1}\Bigg\}(n'-n, m'-m) + \frac{k_{xn'}}{k_{\rho n'm'}}\mathcal{F}\Bigg\{\frac{\partial f}{\partial x}z^{N-l-1}\Bigg\}(n'-n, m'-m)\Bigg]$$
$$\times \epsilon\Bigg[\beta^{(l)}_{n,m}(k_{znm})^{N-l-1} - \frac{k_0}{k_1}\delta^{(l)}_{n,m}(-k_{z1nm})^{N-l-1}\Bigg]\Bigg\}\Bigg), \tag{84}$$

$$\gamma^{(N,r)}_{n',m'} = \frac{k_0}{k_{zn'm'} + k_{z1n'm'}}\Bigg(-\sum_{l=1}^{N-1} \frac{i^{N-l}}{(N-l)!}\sum_m \sum_n \mathcal{F}\{z^{N-l}\}(n'-n, m'-m)$$
$$\times \Bigg\{ c_{n,n'}\Bigg[\alpha^{(l)}_{n,m}(k_{znm})^{N-l}\Bigg(\frac{k_{znm} - k_{zn'm'}}{k_0}\Bigg) + \gamma^{(l)}_{n,m}(-k_{z1nm})^{N-l}\Bigg(\frac{k_{zn'm'} + k_{z1nm}}{k_0}\Bigg)\Bigg]$$
$$+ s_{n,n'}\Bigg[\beta^{(l)}_{n,m}(k_{znm})^{N-l}\Bigg(\frac{-k_0^2 + k_{zn'm'}k_{znm}}{k_0^2}\Bigg) + \delta^{(l)}_{n,m}(-k_{z1nm})^{N-l}\Bigg(\frac{k_1^2 + k_{zn'm'}k_{z1nm}}{k_1 k_0}\Bigg)\Bigg]\Bigg\}$$
$$+ \sum_{l=1}^{N-1} \frac{i^{N-l-1}}{(N-l-1)!}\sum_m \sum_n \frac{k_{\rho nm}}{k_0}\Bigg\{\Bigg[\frac{k_{ym'}}{k_{\rho n'm'}}\mathcal{F}\Bigg\{\frac{\partial f}{\partial x}z^{N-l-1}\Bigg\}(n'-n, m'-m)$$
$$- \frac{k_{xn'}}{k_{\rho n'm'}}\mathcal{F}\Bigg\{\frac{\partial f}{\partial y}z^{N-l-1}\Bigg\}(n'-n, m'-m)\Bigg]\Bigg(\frac{k_{zn'm'}}{k_0}\Bigg)\Bigg[\beta^{(l)}_{n,m}(k_{znm})^{N-l-1} - \frac{k_0}{k_1}\delta^{(l)}_{n,m}(-k_{z1nm})^{N-l-1}\Bigg]$$

$$+ \left[\frac{k_{ym'}}{k_{\rho n'm'}} \mathcal{F}\left\{ \frac{\partial f}{\partial y} z^{N-l-1} \right\}(n'-n, m'-m) + \frac{k_{xn'}}{k_{\rho n'm'}} \mathcal{F}\left\{ \frac{\partial f}{\partial x} z^{N-l-1} \right\}(n'-n, m'-m) \right]$$
$$\times \left. \left. [\alpha_{n,m}^{(l)}(k_{znm})^{N-l-1} - \gamma_{n,m}^{(l)}(-k_{z1nm})^{N-l-1}] \right\} \right), \tag{85}$$

$$\delta_{n',m'}^{(N,r)} = \frac{k_1}{\epsilon k_{zn'm'} + k_{z1n'm'}} \left(-\sum_{l=1}^{N-1} \frac{i^{N-l}}{(N-l)!} \sum_m \sum_n \mathcal{F}\{z^{N-l}\}(n'-n, m'-m) \right.$$
$$\times \left\{ s_{n,n'} \left[\alpha_{n,m}^{(l)}(k_{znm})^{N-l} \left(\frac{k_0^2 - k_{zn'm'}k_{znm}}{k_0^2} \right) - \gamma_{n,m}^{(l)}(-k_{z1nm})^{N-l} \left(\frac{k_0^2 + k_{zn'm'}k_{z1nm}}{k_0^2} \right) \right] \right.$$
$$\left. + c_{n,n'} \left[\beta_{n,m}^{(l)}(k_{znm})^{N-l} \left(\frac{k_{znm} - k_{zn'm'}}{k_0} \right) + \delta_{n,m}^{(l)}(-k_{z1nm})^{N-l} \left(\frac{k_{z1nm} + \epsilon k_{zn'm'}}{k_1} \right) \right] \right\}$$
$$+ \sum_{l=1}^{N-1} \frac{i^{N-l-1}}{(N-l-1)!} \sum_m \sum_n \frac{k_{\rho nm}}{k_0} \left\{ \left[\frac{k_{ym'}}{k_{\rho n'm'}} \mathcal{F}\left\{ \frac{\partial f}{\partial x} z^{N-l-1} \right\}(n'-n, m'-m) \right. \right.$$
$$\left. - \frac{k_{xn'}}{k_{\rho n'm'}} \mathcal{F}\left\{ \frac{\partial f}{\partial y} z^{N-l-1} \right\}(n'-n, m'-m) \right] \left(-\frac{k_{zn'm'}}{k_0} \right) [\alpha_{n,m}^{(l)}(k_{znm})^{N-l-1} - \gamma_{n,m}^{(l)}(-k_{z1nm})^{N-l-1}]$$
$$+ \left[\frac{k_{ym'}}{k_{\rho n'm'}} \mathcal{F}\left\{ \frac{\partial f}{\partial y} z^{N-l-1} \right\}(n'-n, m'-m) + \frac{k_{xn'}}{k_{\rho n'm'}} \mathcal{F}\left\{ \frac{\partial f}{\partial x} z^{N-l-1} \right\}(n'-n, m'-m) \right]$$
$$\left. \left. \times \left[\beta_{n,m}^{(l)}(k_{znm})^{N-l-1} - \frac{k_0}{k_1} \delta_{n,m}^{(l)}(-k_{z1nm})^{N-l-1} \right] \right\} \right), \tag{86}$$

where the (N, r) notation refers to the fact that these are the contributions from the remaining $l > 0$ terms to the field amplitudes at order N. Equations (83)–(86) hold for both horizontally and vertically polarized incident fields. The sum of $\alpha^{(N,0)}$ through $\delta^{(N,0)}$ from Eqs. (58)–(70) for the incident field and $l = 0$ terms with $\alpha^{(N,r)}$ through $\delta^{(N,r)}$ from Eqs. (83)–(86) for the $l > 0$ terms therefore completes the solution for the unknown fields $\alpha^{(N)}$ through $\delta^{(N)}$. The systematic nature of this procedure makes determination of unknown field amplitudes possible up to third or higher order.

Applying the procedure yields solutions for second-order fields, which can be written as

$$\zeta_{n',m'}^{(2)} = \sum_m \sum_n h_{n'-n,m'-m} h_{n,m} g_{\zeta}^{(2)}(k_{xn'}, k_{ym'}, k_{xn}, k_{ym}) \tag{87}$$

where $\zeta = \alpha, \beta, \gamma$, or δ and $g_{\zeta}^{(2)}$ is a corresponding function. For a horizontally polarized incident field,

$$g_{\alpha}^{(2)} = \frac{-2k_{zi}(k_0^2 - k_1^2)}{(k_{zn'm'} + k_{z1n'm'})(k_{zi} + k_{z1i})} \left[c_{n,n'}c_{n,i}(\kappa_{1nm} + k_{z1n'm'}) + s_{n,n'}s_{n,i}(k_{z1n'm'} + \kappa_{2nm}) + \frac{c_{n',i}}{2}(k_{z1i} - k_{z1n'm'}) \right], \tag{88}$$

$$g_{\beta}^{(2)} = \frac{-2k_{zi}(k_0^2 - k_1^2)}{(\epsilon k_{zn'm'} + k_{z1n'm'})(k_{zi} + k_{z1i})} \left[s_{n,n'}c_{n,i} \left(\epsilon k_0 + \frac{k_{z1n'm'}}{k_0} \kappa_{1nm} \right) - c_{n,n'}s_{n,i} \left(\epsilon k_0 + \frac{k_{z1n'm'}}{k_0} \kappa_{2nm} \right) \right.$$
$$\left. + s_{n,i} \frac{\epsilon k_{\rho n'm'}}{k_0} \kappa_{3nm} + \frac{s_{n',i}}{2} \left(\epsilon k_0 - \frac{k_{z1n'm'}}{k_0} k_{z1i} \right) \right], \tag{89}$$

$$g_{\gamma}^{(2)} = \frac{-2k_{zi}(k_0^2 - k_1^2)}{(k_{zn'm'} + k_{z1n'm'})(k_{zi} + k_{z1i})} \left[c_{n,n'}c_{n,i}(\kappa_{1nm} - k_{zn'm'}) + s_{n,n'}s_{n,i}(-k_{zn'm'} + \kappa_{2nm}) + \frac{c_{n',i}}{2}(k_{z1i} + k_{zn'm'}) \right], \tag{90}$$

$$g_{\delta}^{(2)} = \frac{-2k_{zi}(k_0^2 - k_1^2)\sqrt{\epsilon}}{(\epsilon k_{zn'm'} + k_{z1n'm'})(k_{zi} + k_{z1i})} \left[s_{n,n'}c_{n,i} \left(k_0 - \frac{k_{zn'm'}}{k_0} \kappa_{1nm} \right) - c_{n,n'}s_{n,i} \left(k_0 - \frac{k_{zn'm'}}{k_0} \kappa_{2nm} \right) + s_{n,i} \frac{k_{\rho n'm'}}{k_0} \kappa_{3nm} \right.$$
$$\left. + \frac{s_{n',i}}{2} \left(k_0 + \frac{k_{zn'm'}}{k_0} k_{z1i} \right) \right], \tag{91}$$

where

$$\kappa_{1nm} = k_{znm} - k_{z1nm}, \tag{92}$$

$$\kappa_{2nm} = \frac{k_{znm}k_{z1nm}(k_0^2 - k_1^2)}{k_0^2(\epsilon k_{znm} + k_{z1nm})}, \tag{93}$$

$$\kappa_{3nm} = \frac{k_{\rho nm}k_0^2}{k_{\rho nm}^2 + k_{znm}k_{z1nm}}. \tag{94}$$

For a vertically polarized incident field,

$$g_\alpha^{(2)} = \frac{-2k_{zi}(k_0^2 - k_1^2)}{(k_{zn'm'} + k_{z1n'm'})(\epsilon k_{zi} + k_{z1i})}\left[-c_{n,n'}s_{n,i}\left(\frac{k_{z1i}}{k_0}\right)(\kappa_{1nm} + k_{z1n'm'}) + s_{n,n'}c_{n,i}\left(\frac{k_{z1i}}{k_0}\right)(k_{z1n'm'} + \kappa_{2nm}) - s_{n,n'}\left(\frac{\epsilon k_{\rho i}}{k_0}\right)\kappa_{3nm} - \frac{s_{n',i}}{2}\left(\epsilon k_0 - \frac{k_{z1n'm'}}{k_0}k_{z1i}\right)\right], \tag{95}$$

$$g_\beta^{(2)} = \frac{-2k_{zi}(k_0^2 - k_1^2)}{(\epsilon k_{zn'm'} + k_{z1n'm'})(\epsilon k_{zi} + k_{z1i})}\left[-s_{n,n'}s_{n,i}\left(\frac{k_{z1i}}{k_0}\right)\left(\epsilon k_0 + \frac{k_{z1n'm'}}{k_0}\kappa_{1nm}\right) - c_{n,n'}c_{n,i}\left(\frac{k_{z1i}}{k_0}\right)\left(\epsilon k_0 + \frac{k_{z1n'm'}}{k_0}\kappa_{2nm}\right) + c_{n,n'}\left(\frac{\epsilon k_{\rho i}k_{z1n'm'}}{k_0^2}\right)\kappa_{3nm} + \left(\frac{\epsilon k_{\rho n'm'}\kappa_{3nm}}{k_0^2}\right)\left(k_{z1i}c_{n,i} + \frac{k_{\rho nm}k_{\rho i}}{k_0^2}\kappa_{1nm}\right) + \frac{c_{n',i}}{2}(\epsilon k_{z1i} - \epsilon k_{z1n'm'})\right], \tag{96}$$

$$g_\gamma^{(2)} = \frac{-2k_{zi}(k_0^2 - k_1^2)}{(k_{zn'm'} + k_{z1n'm'})(\epsilon k_{zi} + k_{z1i})}\left[-c_{n,n'}s_{n,i}\left(\frac{k_{z1i}}{k_0}\right)(\kappa_{1nm} - k_{zn'm'}) + s_{n,n'}c_{n,i}\left(\frac{k_{z1i}}{k_0}\right)(-k_{zn'm'} + \kappa_{2nm}) - s_{n,n'}\left(\frac{\epsilon k_{\rho i}}{k_0}\right)\kappa_{3nm} - \frac{s_{n',i}}{2}\left(\epsilon k_0 + \frac{k_{zn'm'}}{k_0}k_{z1i}\right)\right], \tag{97}$$

$$g_\delta^{(2)} = \frac{-2k_{zi}(k_0^2 - k_1^2)\sqrt{\epsilon}}{(\epsilon k_{zn'm'} + k_{z1n'm'})(\epsilon k_{zi} + k_{z1i})}\left[-s_{n,n'}s_{n,i}\left(\frac{k_{z1i}}{k_0}\right)\left(k_0 - \frac{k_{zn'm'}}{k_0}\kappa_{1nm}\right) - c_{n,n'}c_{n,i}\left(\frac{k_{z1i}}{k_0}\right)\left(k_0 - \frac{k_{zn'm'}}{k_0}\kappa_{2nm}\right) - c_{n,n'}\left(\frac{\epsilon k_{\rho i}k_{zn'm'}}{k_0^2}\right)\kappa_{3nm} + \left(\frac{k_{\rho n'm'}\kappa_{3nm}}{k_0^2}\right)\left(k_{z1i}c_{n,i} + \frac{k_{\rho nm}k_{\rho i}}{k_0^2}\kappa_{1nm}\right) + \frac{c_{n',i}}{2}(k_{z1i} + \epsilon k_{zn'm'})\right]. \tag{98}$$

Equations (88)–(89) and (95)–(96) reduce to the second-order specular reflection coefficient corrections described in Ref. 13 when $n' = 0$ and $m' = 0$, except for a minus sign difference in cross-polarized terms due to differing coordinate systems.

6. THIRD-ORDER SOLUTION

The solution for third-order fields proceeds similarly. Equations (58)–(70) with $N = 3$ yield the incident field and $l = 0$ terms, and Eqs. (83)–(86) with $N = 3$ are used to obtain the $l = 1$ and $l = 2$ terms. Solutions are found to be of the form

$$\zeta_{n',m'}^{(3)} = \sum_m \sum_n \sum_{m_1} \sum_{n_1} h_{n,m}h_{n_1,m_1}h_{n'-n-n_1,m'-m-m_1}g_\zeta^{(3)}(k_{xn'}, k_{ym'}, k_{xn}, k_{ym}, k_{xn_1}, k_{ym_1}), \tag{99}$$

where $\zeta = \alpha$, β, γ, or δ and $g_\zeta^{(3)}$ is a corresponding function. For a horizontally polarized incident field,

$$g_\alpha^{(3)} = \left(\frac{i}{k_{zm'n'} + k_{z1n'm'}}\right)\left\{\left(\frac{k_{zi}(k_0^2 - k_1^2)}{k_{zi} + k_{z1i}}\right)\left[-c_{n,n'}c_{n,i}(k_{z1n'm'}\kappa_{1nm} + k_0\kappa_{4nm}) - s_{n,n'}s_{n,i}\kappa_{2nm}(\kappa_{5nm} + k_{z1n'm'}) + (2c_{n,i}k_{\rho nm} - k_{\rho i})(k_{\rho n'm'} - k_{\rho n''m''}c_{n',n''}) - \frac{c_{n',i}}{3}(k_{z1n'm'}k_{z1i} - k_{zi}^2 - k_{z1i}^2)\right] + k_{\rho n'm'}k_{\rho n''m''}(g_\alpha^{(2)} - g_\gamma^{(2)}) + c_{n',n''}(k_0G_{1h}^{(2)} - k_{z1n'm'}G_{2h}^{(2)}) - s_{n',n''}(k_0G_{3h}^{(2)} + k_{z1n'm'}G_{4h}^{(2)})\right\}, \tag{100}$$

$$g_\beta^{(3)} = \left(\frac{-i}{\epsilon k_{zm'n'} + k_{z1n'm'}}\right)\left\{\left(\frac{k_{zi}(k_0^2 - k_1^2)}{k_{zi} + k_{z1i}}\right)\left[s_{n,n'}c_{n,i}(\epsilon k_0\kappa_{1nm} + k_{z1n'm'}\kappa_{4nm}) - c_{n,n'}s_{n,i}\kappa_{2nm}\left(\epsilon k_0 + \frac{k_{z1n'm'}}{k_0}\kappa_{5nm}\right) - s_{n',n''}(2c_{n,i}k_{\rho nm} - k_{\rho i})\left(\frac{k_{\rho n''m''}k_{z1n'm'}}{k_0}\right) - \frac{s_{n',i}}{3}\left(\frac{k_1^2k_{z1i} - k_{z1n'm'}(k_{zi}^2 + k_{z1i}^2)}{k_0}\right)\right] - \epsilon k_{\rho n'm'}k_{\rho n''m''}(g_\beta^{(2)} - g_\delta^{(2)}/\sqrt{\epsilon}) + s_{n',n''}(k_{z1n'm'}G_{1h}^{(2)} - \epsilon k_0G_{2h}^{(2)}) + c_{n',n''}(k_{z1n'm'}G_{3h}^{(2)} + \epsilon k_0G_{4h}^{(2)})\right\}, \tag{101}$$

$$g_{\gamma}^{(3)} = \left(\frac{-i}{k_{zm'n'} + k_{z1n'm'}}\right)\left\{\left(\frac{k_{zi}(k_0^2 - k_1^2)}{k_{zi} + k_{z1i}}\right)\left[-c_{n,n'}c_{n,i}(k_{zn'm'}\kappa_{1nm} - k_0\kappa_{4nm}) - s_{n,n'}s_{n,i}\kappa_{2nm}(-\kappa_{5nm} + k_{zn'm'})\right.\right.$$
$$\left. - (2c_{n,i}k_{\rho nm} - k_{\rho i})(k_{\rho n'm'} - k_{\rho n''m''}c_{n',n''}) - \frac{c_{n',i}}{3}(k_{zn'm'}k_{z1i} + k_{zi}^2 + k_{z1i}^2)\right] - k_{\rho n'm'}k_{\rho n''m''}(g_{\alpha}^{(2)} - g_{\gamma}^{(2)})$$
$$\left. + c_{n',n''}(-k_0 G_{1h}^{(2)} - k_{zn'm'}G_{2h}^{(2)}) - s_{n',n''}(-k_0 G_{3h}^{(2)} + k_{zn'm'}G_{4h}^{(2)})\right\}, \tag{102}$$

$$g_{\delta}^{(3)} = \left(\frac{-i\sqrt{\epsilon}}{\epsilon k_{zm'n'} + k_{z1n'm'}}\right)\left\{\left(\frac{k_{zi}(k_0^2 - k_1^2)}{k_{zi} + k_{z1i}}\right)\left[s_{n,n'}c_{n,i}(k_0\kappa_{1nm} - k_{zn'm'}\kappa_{4nm}) - c_{n,n'}s_{n,i}\kappa_{2nm}\left(k_0 - \frac{k_{zn'm'}}{k_0}\kappa_{5nm}\right)\right.\right.$$
$$\left. + s_{n',n''}(2c_{n,i}k_{\rho nm} - k_{\rho i})\left(\frac{k_{\rho n''m''}k_{zn'm'}}{k_0}\right) - \frac{s_{n',i}}{3}\left(\frac{k_0^2 k_{z1i} + k_{zn'm'}(k_{zi}^2 + k_{z1i}^2)}{k_0}\right)\right]$$
$$\left. - k_{\rho n'm'}k_{\rho n''m''}(g_{\beta}^{(2)} - g_{\delta}^{(2)}/\sqrt{\epsilon}) + s_{n',n''}(-k_{zn'm'}G_{1h}^{(2)} - k_0 G_{2h}^{(2)}) + c_{n',n''}(-k_{zn'm'}G_{3h}^{(2)} + k_0 G_{4h}^{(2)})\right\}, \tag{103}$$

where $n'' = n + n_1$, $m'' = m + m_1$,

$$\kappa_{4nm} = \frac{k_{znm}^2 - k_{znm}k_{z1nm} + k_{z1nm}^2}{k_0}, \tag{104}$$

$$\kappa_{5nm} = \frac{k_{znm} + \epsilon k_{z1nm}}{1 - \epsilon}, \tag{105}$$

$$G_{1h}^{(2)} = k_0(\epsilon g_{\gamma}^{(2)} - g_{\alpha}^{(2)}), \tag{106}$$

$$G_{2h}^{(2)} = k_{zn''m''}g_{\alpha}^{(2)} + k_{z1n''m''}g_{\gamma}^{(2)}, \tag{107}$$

$$G_{3h}^{(2)} = k_{zn''m''}g_{\beta}^{(2)} + \sqrt{\epsilon}k_{z1n''m''}g_{\delta}^{(2)}, \tag{108}$$

$$G_{4h}^{(2)} = k_0(g_{\beta}^{(2)} - \sqrt{\epsilon}g_{\delta}^{(2)}), \tag{109}$$

and the $g_{\zeta}^{(2)}$ functions referenced are for horizontal incidence and are evaluated with arguments

$$g_{\zeta}^{(2)}(k_{xn} + k_{xn_1} - k_{xi}, k_{ym} + k_{ym_1} - k_{yi}, k_{xn_1}, k_{ym_1}) \tag{110}$$

as specified in Eqs. (87)–(91).

For a vertically polarized incident field,

$$g_{\alpha}^{(3)} = \left(\frac{i}{k_{zm'n'} + k_{z1n'm'}}\right)\left\{\left(\frac{k_{zi}(k_0^2 - k_1^2)}{\epsilon k_{zi} + k_{z1i}}\right)\left[c_{n,n'}s_{n,i}\left(\frac{k_{z1i}}{k_0}\right)(k_{z1n'm'}\kappa_{1nm} + k_0\kappa_{4nm})\right.\right.$$
$$- s_{n,n'}c_{n,i}\kappa_{2nm}\left(\frac{k_{z1i}}{k_0}\right)(\kappa_{5nm} + k_{z1n'm'}) + s_{n,n'}\left(\frac{k_{\rho i}k_{\rho nm}}{k_0}\right)\left(\frac{k_1^2(k_0^2 - k_1^2) + k_{z1n'm'}(\epsilon k_{znm}^3 + k_{z1nm}^3)}{k_0^2(\epsilon k_{znm} + k_{z1nm})}\right)$$
$$- 2s_{n,i}\left(\frac{k_{z1i}k_{\rho nm}}{k_0}\right)(k_{\rho n'm'} - k_{\rho n''m''}c_{n',n''}) - s_{n',n''}k_{\rho n''m''}\left(\frac{k_{z1n'm'}k_{\rho i}k_{\rho nm}^2}{k_0^3}\right)$$
$$\left. + \frac{s_{n',i}k_0}{3(k_0^2 - k_1^2)}\left(\frac{k_{z1n'm'}(\epsilon k_{zi}^4 - k_{z1i}^4)}{k_0^2} + k_{z1i}(\epsilon k_{z1i}^2 - k_{zi}^2)\right)\right]$$
$$\left. + k_{\rho n'm'}k_{\rho n''m''}(g_{a}^{(2)} - g_{\gamma}^{(2)}) + c_{n',n''}(k_0 G_{1v}^{(2)} - k_{z1n'm'}G_{2v}^{(2)}) - s_{n',n''}(k_0 G_{3v}^{(2)} + k_{z1n'm'}G_{4v}^{(2)})\right\}, \tag{111}$$

$$g_{\beta}^{(3)} = \left(\frac{-i}{\epsilon k_{zm'n'} + k_{z1n'm'}}\right)\left\{\left(\frac{k_{zi}(k_0^2 - k_1^2)}{\epsilon k_{zi} + k_{z1i}}\right)\left[-s_{n,n'}s_{n,i}\left(\frac{k_{z1i}}{k_0}\right)(\epsilon k_0\kappa_{1nm} + k_{z1n'm'}\kappa_{4nm})\right.\right.$$
$$- c_{n,n'}c_{n,i}\kappa_{2nm}\left(\frac{k_{z1i}}{k_0}\right)\left(\epsilon k_0 + \frac{k_{z1n'm'}}{k_0}\kappa_{5nm}\right) + c_{n,n'}\left(\frac{\epsilon k_{\rho i}k_{\rho nm}}{k_0^2}\right)\left(\frac{\epsilon k_{znm}^3 + k_{z1nm}^3 + k_{z1n'm'}(k_0^2 - k_1^2)}{\epsilon k_{znm} + k_{z1nm}}\right)$$

$$+ 2s_{n,i}s_{n',n''}\left(\frac{k_{\rho nm}k_{\rho n''m''}k_{z1n'm'}k_{z1i}}{k_0^2}\right) - \left(\frac{\epsilon k_{\rho nm}^2 k_{\rho i}}{k_0^2}\right)(k_{\rho n'm'} - c_{n',n''}k_{\rho n''m''})$$

$$- \frac{c_{n',i}}{3(k_0^2 - k_1^2)}(\epsilon(\epsilon k_{zi}^4 - k_{z1i}^4) + k_{z1n'm'}k_{z1i}(\epsilon k_{z1i}^2 - k_{zi}^2))\Bigg] - \epsilon k_{\rho n'm'}k_{\rho n''m''}(g_\beta^{(2)} - g_\delta^{(2)}/\sqrt{\epsilon})$$

$$+ s_{n',n''}(k_{z1n'm'}G_{1v}^{(2)} - \epsilon k_0 G_{2v}^{(2)}) + c_{n',n''}(k_{z1n'm'}G_{3v}^{(2)} + \epsilon k_0 G_{4v}^{(2)})\Bigg\}, \quad (112)$$

$$g_\gamma^{(3)} = \left(\frac{-i}{k_{zm'n'} + k_{z1n'm'}}\right)\Bigg\{\left(\frac{k_{zi}(k_0^2 - k_1^2)}{\epsilon k_{zi} + k_{z1i}}\right)\Bigg[c_{n,n'}s_{n,i}\left(\frac{k_{z1i}}{k_0}\right)(k_{zn'm'}\kappa_{1nm} - k_0\kappa_{4nm})$$

$$- s_{n,n'}c_{n,i}\kappa_{2nm}\left(\frac{k_{z1i}}{k_0}\right)(-\kappa_{5nm} + k_{zn'm'}) + s_{n,n'}\left(\frac{k_{\rho i}k_{\rho nm}}{k_0}\right)\left(\frac{k_1^2(k_1^2 - k_0^2) + k_{zn'm'}(\epsilon k_{znm}^3 + k_{z1nm}^3)}{k_0^2(\epsilon k_{znm} + k_{z1nm})}\right)$$

$$+ 2s_{n,i}\left(\frac{k_{z1i}k_{\rho nm}}{k_0}\right)(k_{\rho n'm'} - k_{\rho n''m''}c_{n',n''}) - s_{n',n''}k_{\rho n''m''}\left(\frac{k_{zn'm'}k_{\rho i}k_{\rho nm}^2}{k_0^3}\right)$$

$$+ \frac{s_{n',i}k_0}{3(k_0^2 - k_1^2)}\left(\frac{k_{zn'm'}(\epsilon k_{zi}^4 - k_{z1i}^4)}{k_0^2} - k_{z1i}(\epsilon k_{z1i}^2 - k_{zi}^2)\right)\Bigg]$$

$$- k_{\rho n'm'}k_{\rho n''m''}(g_\alpha^{(2)} - g_\gamma^{(2)}) + c_{n',n''}(-k_0 G_{1v}^{(2)} - k_{zn'm'}G_{2v}^{(2)}) - s_{n',n''}(-k_0 G_{3v}^{(2)} + k_{zn'm'}G_{4v}^{(2)})\Bigg\}, \quad (113)$$

$$g_\delta^{(3)} = \left(\frac{-i\sqrt{\epsilon}}{\epsilon k_{zm'n'} + k_{z1n'm'}}\right)\Bigg\{\left(\frac{k_{zi}(k_0^2 - k_1^2)}{\epsilon k_{zi} + k_{z1i}}\right)\Bigg[-s_{n,n'}s_{n,i}\left(\frac{k_{z1i}}{k_0}\right)(k_0\kappa_{1nm} - k_{zn'm'}\kappa_{4nm})$$

$$- c_{n,n'}c_{n,i}\kappa_{2nm}\left(\frac{k_{z1i}}{k_0}\right)\left(k_0 - \frac{k_{zn'm'}}{k_0}\kappa_{5nm}\right) + c_{n,n'}\left(\frac{k_{\rho i}k_{\rho nm}}{k_0^2}\right)\left(\frac{\epsilon k_{znm}^3 + k_{z1nm}^3 - \epsilon k_{zn'm'}(k_0^2 - k_1^2)}{\epsilon k_{znm} + k_{z1nm}}\right)$$

$$- 2s_{n,i}s_{n',n''}\left(\frac{k_{\rho nm}k_{\rho n''m''}k_{zn'm'}k_{z1i}}{k_0^2}\right) - \left(\frac{k_{\rho nm}^2 k_{\rho i}}{k_0^2}\right)(k_{\rho n'm'} - c_{n',n''}k_{\rho n''m''})$$

$$- \frac{c_{n',i}}{3(k_0^2 - k_1^2)}(\epsilon k_{zi}^4 - k_{z1i}^4 - k_{zn'm'}k_{z1i}(\epsilon k_{z1i}^2 - k_{zi}^2))\Bigg] - k_{\rho n'm'}k_{\rho n''m''}(g_\beta^{(2)} - g_\delta^{(2)}/\sqrt{\epsilon})$$

$$- s_{n',n''}(k_{zn'm'}G_{1v}^{(2)} + k_0 G_{2v}^{(2)}) + c_{n',n''}(-k_{zn'm'}G_{3v}^{(2)} + k_0 G_{4v}^{(2)})\Bigg\}, \quad (114)$$

where the G_{1v} through G_{4v} functions are defined analogously to those for horizontal polarization except that the $g_\zeta^{(2)}$ functions are for vertical incidence and are evaluated with arguments

$$g_\zeta^{(2)}(k_{xn} + k_{xn_1} - k_{xi}, k_{ym} + k_{ym_1} - k_{yi}, k_{xn_1}, k_{ym_1}) \quad (115)$$

as specified in Eqs. (95)–(98).

7. REFLECTED AND TRANSMITTED POWER

Given the field solution to third order in surface height, reflected and transmitted powers can also be derived to third order. Since the power in a particular Floquet mode is directly proportional to its amplitude squared, and since distinct polarizations are orthogonal, the relevant quantities to consider are

$$|\zeta_{n',m'}|^2 = |\zeta_{n',m'}^{(0)} + \zeta_{n',m'}^{(1)} + \zeta_{n',m'}^{(2)} + \zeta_{n',m'}^{(3)} + \cdots|^2, \quad (116)$$

where ζ represents α, β, γ, or δ. Collecting terms of identical order yields

$$\begin{aligned}|\zeta_{n',m'}|^2 = &(|\zeta_{n',m'}^{(0)}|^2) + (2\,\mathrm{Re}\{\zeta_{n',m'}^{(0)*}\zeta_{n',m'}^{(1)}\})\\ &+ (|\zeta_{n',m'}^{(1)}|^2 + 2\,\mathrm{Re}\{\zeta_{n',m'}^{(0)*}\zeta_{n',m'}^{(2)}\})\\ &+ (2\,\mathrm{Re}\{\zeta_{n',m'}^{(1)*}\zeta_{n',m'}^{(2)}\} + 2\,\mathrm{Re}\{\zeta_{n',m'}^{(0)*}\zeta_{n',m'}^{(3)}\})\\ &+ (|\zeta_{n',m'}^{(2)}|^2 + 2\,\mathrm{Re}\{\zeta_{n',m'}^{(1)*}\zeta_{n',m'}^{(3)}\}\\ &+ 2\,\mathrm{Re}\{\zeta_{n',m'}^{(0)*}\zeta_{n',m'}^{(4)}\}) + \cdots\end{aligned} \quad (117)$$

where individual orders are grouped inside parentheses, and a fourth-order term has been included as well, even though $\zeta^{(4)}$ has not yet been derived. Immediately it can be recognized that the zeroth-order term represents the reflectivity of a flat surface, and also that terms multiplying $\zeta_{n',m'}^{(0)}$ are evaluated only with $n' = m' = 0$ since $\zeta_{n',m'}^{(0)}$ vanishes for all other indices; these terms represent corrections to the flat-surface reflectivity. If it is assumed that the surface has a zero spatial average value (i.e., $h_{0,0} = 0$) then the first-order term vanishes since it is directly proportional to $h_{0,0}$. All other terms exist in the general case and contribute to reflected and transmitted powers. Fractions of the incident power reflected into a specific polarization of a Floquet mode (n', m') can be shown to be

$$\mathrm{Re}\left\{\frac{k_{zn'm'}}{k_{zi}}\right\}|\zeta|^2, \tag{118}$$

where $\zeta = \alpha$ or β, and the fraction of power transmitted into a specific polarization of Floquet mode (n', m') in a lossless medium can be shown to be

$$\mathrm{Re}\left\{\frac{k_{z1n'm'}}{k_{zi}}\right\}|\zeta|^2, \tag{119}$$

where $\zeta = \gamma$ or δ.

Since the small-perturbation method is frequently applied in the analysis of stochastic surfaces, it is also of interest to consider scattered and transmitted coherent and incoherent powers. In this case, the results are considerably simplified by assuming that each point on the surface profile $z(x, y)$ is a zero mean random variable [i.e., $\langle z(x, y)\rangle = 0$] so that $\langle h_{n',m'}\rangle = 0$ also, as shown in Eq. (2).

In this case, the coherent reflectivity,

$$|\langle\zeta\rangle|^2 = |\langle\Gamma_\zeta^{\mathrm{eff}}\rangle|^2 \tag{120}$$

with $\zeta = \alpha$ or β is found to exist only in the specular direction $n' = m' = 0$ up to third order, and the effective reflection coefficient $\langle\Gamma_\zeta^{\mathrm{eff}}\rangle$ is given by

$$\begin{aligned}\langle\Gamma_\zeta^{\mathrm{eff}}\rangle = \Gamma_\zeta &+ \sum_m \sum_n \langle|h_{n,m}|^2\rangle g_\zeta^{(2)}(k_{xi}, k_{yi}, k_{xn}, k_{ym}) \\ &+ \sum_m \sum_n \sum_{m_1} \sum_{n_1} \langle h_{n,m} h_{n_1,m_1} h_{-n-n_1,-m-m_1}\rangle \\ &\times g_\zeta^{(3)}(k_{xi}, k_{yi}, k_{xn}, k_{ym}, k_{xn_1}, k_{ym_1})\end{aligned} \tag{121}$$

to third order in surface height. A corresponding equation can be derived for the effective transmission coefficient in a lossless medium. Note an expansion of the coherent power $|\langle\Gamma_\zeta^{\mathrm{eff}}\rangle|^2$ similar to equation (117) is required to group coherent power terms to third order consistently. Note also that the second and third order terms above involve the surface power spectrum $\langle|h_{n,m}|^2\rangle$ and the surface bi-spectrum $\langle h_{n,m} h_{n_1,m_1} h_{-n-n_1,-m-m_1}\rangle$ respectively. If it is further assumed that the surface is a Gaussian random process, the bispectrum vanishes and there is no third-order contribution to the effective reflection coefficient. Calculation of the fourth-order coherent reflected power requires knowledge of $\zeta_{0,0}^{(4)}$ and is not considered here.

The expansion for incoherent powers produces

$$\begin{aligned}&\langle|\zeta_{n',m'} - \langle\zeta_{n',m'}\rangle|^2\rangle \\ &\quad = \langle|\zeta_{n',m'}^{(1)}|^2\rangle + 2\,\mathrm{Re}\{\langle\zeta_{n',m'}^{(1)*}\zeta_{n',m'}^{(2)}\rangle\} \\ &\qquad + \langle|\zeta_{n',m'}^{(2)} - \langle\zeta_{n',m'}^{(2)}\rangle|^2\rangle + 2\,\mathrm{Re}\{\langle\zeta_{n',m'}^{(1)*}\zeta_{n',m'}^{(3)}\rangle\}\end{aligned} \tag{122}$$

to fourth order; note that the third-order solution for fields is sufficient to determine incoherent scattered and transmitted powers to fourth order. Equation (122) can be rewritten as

$$\begin{aligned}\langle|\zeta_{n',m'} - \langle\zeta_{n',m'}\rangle|\rangle^2 = {}& |g_\zeta^{(1)}(k_{xn'}, k_{ym'})|^2 \langle|h_{n',m'}|^2\rangle \\ &+ 2\,\mathrm{Re}\left\{\sum_m \sum_n \langle h_{n',m'}^* h_{n,m} h_{n'-n,m'-m}\rangle g_\zeta^{(1)*}(k_{xn'}, k_{ym'}) g_\zeta^{(2)}(k_{xn'}, k_{ym'}, k_{xn}, k_{ym})\right\} \\ &+ \sum_m \sum_n \sum_{m_1} \sum_{n_1} [\langle h_{n'-n,m'-m} h_{n,m} h_{n'-n_1,m'-m_1}^* h_{n_1,m_1}^*\rangle - \langle h_{n'-n,m'-m} h_{n,m}\rangle \\ &\times \langle h_{n'-n_1,m'-m_1}^* h_{n_1,m_1}^*\rangle] g_\zeta^{(2)}(k_{xn'}, k_{ym'}, k_{xn}, k_{ym}) g_\zeta^{(2)*}(k_{xn'}, k_{ym'}, k_{xn_1}, k_{ym_1}) \\ &+ 2\,\mathrm{Re}\left\{\sum_m \sum_n \sum_{m_1} \sum_{n_1} \langle h_{n,m} h_{n_1,m_1} h_{n'-n-n_1,m'-m-m_1} h_{-n',-m'}\rangle\right. \\ &\left.\times\, g_\zeta^{(3)}(k_{xn'}, k_{ym'}, k_{xn}, k_{ym}, k_{xn_1}, k_{ym_1}) g_\zeta^{(1)*}(k_{xn'}, k_{ym'})\right\},\end{aligned} \tag{123}$$

showing the dependencies of incoherent power at second order on the surface spectrum, at third order on the surface bispectrum, and at fourth order on quantities that can be related to the surface trispectrum, power spectrum, and correlations between Fourier coefficients. Again for a Gaussian random process the bispectrum and third-order power terms vanish, while the fourth order power term can be expressed in terms of the surface power spectrum only.[2]

Incoherent scattering cross sections per unit area for a nonperiodic surface (whose dimensions must be large compared with the electromagnetic wavelength and any surface features, and neglecting edge scattering effects) can also be derived from these results by considering the limit as the surface periods approach infinity following Ref. 2. The scattering cross section per unit area at a particular scattering angle [related to (n', m')] and in a particular polarization can be shown to be

$$\sigma_\zeta = 4\pi k_0^2 \cos^2\theta_s \frac{\langle|\zeta|^2\rangle}{\delta k_x \delta k_y}, \tag{124}$$

where $\delta k_x = (2\pi)/L_x$ and $\delta k_y = (2\pi)/L_y$ are differential quantities that cancel when $h_{n',m'}$ terms are related to their continuous counterparts.

The definitions

$$\frac{\langle|h_{n',m'}|^2\rangle}{\delta k_x \delta k_y} = W(k_{xn'} - k_{xi}, k_{ym'} - k_{yi}), \tag{125}$$

$$\frac{\langle h_{n',m'} h_{n,m} h_{-n'-n,-m'-m}\rangle}{(\delta k_x)^2(\delta k_y)^2} = B(k_{xn'} - k_{xi}, k_{ym'} - k_{yi}, k_{xn} - k_{xi}, k_{ym} - k_{yi}), \tag{126}$$

and

$$\frac{\langle h_{n',m'} h_{n,m} h_{n_1,m_1} h_{-n'-n-n_1,-m'-m-m_1}\rangle}{(\delta k_x)^3(\delta k_y)^3} = T(k_{xn'} - k_{xi}, k_{ym'} - k_{yi}, k_{xn} - k_{xi}, k_{ym} - k_{yi}, k_{xn_1} - k_{xi}, k_{ym_1} - k_{yi}), \tag{127}$$

where W, B, and T represent the continuous surface power spectrum, bispectrum, and a quantity that can be related to the trispectrum, respectively, allow the sums over n and m variables in the coherent and the incoherent power expressions to be converted into integrals over the corresponding wave numbers. For a continuous Gaussian random process, σ_ζ up to fourth order can be simplified to

$$\begin{aligned}\sigma_\zeta(k_{xn'}, k_{ym'}) = 4\pi k_0^2 \cos^2\theta_s \Bigg(& |g_\zeta^{(1)}(k_{xn'}, k_{ym'})|^2 W(k_{xn'} - k_{xi}, k_{ym'} - k_{yi}) \\ & + \int_{-\infty}^{\infty} dk_{xn} \int_{-\infty}^{\infty} dk_{ym} \{W(k_{xn} - k_{xi}, k_{ym} - k_{yi}) W(k_{xn'} - k_{xn}, k_{ym'} - k_{ym})[|g_\zeta^{(2)}(k_{xn'}, k_{ym'}, k_{xn}, k_{ym})|^2 \\ & + g_\zeta^{(2)}(k_{xn'}, k_{ym'}, k_{xn}, k_{ym}) g_\zeta^{(2)*}(k_{xn'}, k_{ym'}, k_{xn'} - k_{xn} + k_{xi}, k_{ym'} - k_{ym} + k_{yi})]\} \\ & + 2\,\mathrm{Re}\Bigg\{ W(k_{xn'} - k_{xi}, k_{ym'} - k_{yi}) g_\zeta^{(1)*}(k_{xn'}, k_{ym'}) \int_{-\infty}^{\infty} \mathrm{d}k_{xn} \int_{-\infty}^{\infty} \mathrm{d}k_{ym} W(k_{xn} - k_{xi}, k_{ym} - k_{yi}) \\ & \times [g_\zeta^{(3)}(k_{xn'}, k_{ym'}, k_{xn}, k_{ym}, k_{xn'}, k_{ym'}) + g_\zeta^{(3)}(k_{xn'}, k_{ym'}, k_{xn'}, k_{ym'}, k_{xn}, k_{ym}) \\ & \times g_\zeta^{(3)}(k_{xn'}, k_{ym'}, k_{xn}, k_{ym}, 2k_{xi} - k_{xn}, 2k_{yi} - k_{ym})]\Bigg\}\Bigg). \end{aligned} \tag{128}$$

The first line of Eq. (128) represents the second-order $\zeta^{(1)}\zeta^{(1)*}$ term, and the following two integrals are the fourth-order $\zeta^{(2)}\zeta^{(2)*}$ (considered previously in Ref. 2) and $\zeta^{(3)}\zeta^{(1)*}$ (not considered previously) terms, respectively. Note that the $\zeta^{(3)}\zeta^{(1)*}$ term is directly proportional to $g_\zeta^{(1)*}(k_{xn'}, k_{ym'}) W(k_{xn'} - k_{xi}, k_{ym'} - k_{yi})$ and vanishes whenever the $\zeta^{(1)}\zeta^{(1)*}$ term vanishes. Since cross-polarized $g_\zeta^{(1)}$ vanishes in the plane of incidence, the $\zeta^{(2)}\zeta^{(2)*}$ term alone is sufficient for calculation of cross-polarized backscattering as in Ref. 2.

A final quantity of interest is the total fraction of power reflected from a surface in all scattered polarizations, which is defined as the total surface reflectivity and which can be related to the surface emissivity to determine surface thermal emission. It has been shown in Ref. 15 that the small-perturbation method produces an expansion in surface slope and not in surface height for this quantity, making the preceding equations sufficient for determining the surface total reflectivity up to third order in surface slope. Calculation of the total surface reflectivity requires inclusion of both coherent and incoherent terms, and powers in all scattered Floquet modes are summed. Application of these results to the computation of surface thermal emission will be discussed in a future paper.

8. SAMPLE RESULTS

Example SPM results for both periodic and nonperiodic surfaces are considered in this section. Results will be compared at second, third, and fourth order to determine the influence of higher-order terms, and fourth-order predictions are also compared with predictions of other scattering theories: the small-slope approximation[12] (SSA) and a Monte Carlo simulation with the method of moments[19,20] (MOM) in the nonperiodic case, and the extended boundary condition (EBC) numerical method[21] in the periodic-surface case.

The first surface type considered is a nonperiodic, Gaussian random-process surface with an isotropic Gaussian correlation function, completely characterized by the rms surface height h and correlation length l parameters. Again for this surface type, third-order power terms vanish, and Eq. (128) was used to calculate cross sections per unit area up to fourth order. Fourth-order terms required a numerical evaluation of the integrals in Eq. (128), which was performed with Gauss–Legendre

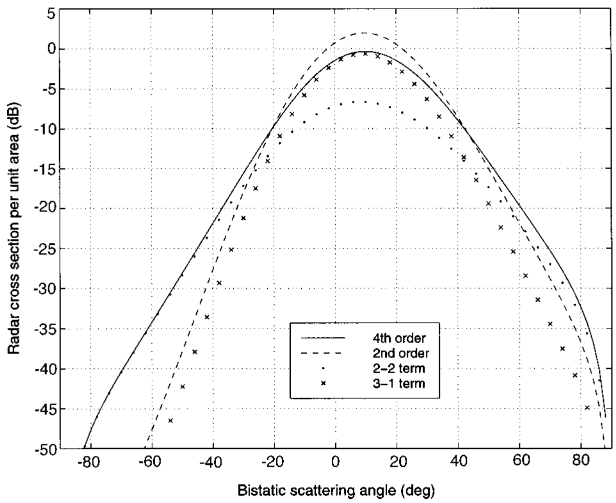

Fig. 1. In plane, *HH* bistatic scattering cross sections per unit area for a Gaussian correlation function surface with $h = 0.06\lambda$ and $l = \lambda$, $\theta_i = 10°$, and $\epsilon = 3$.

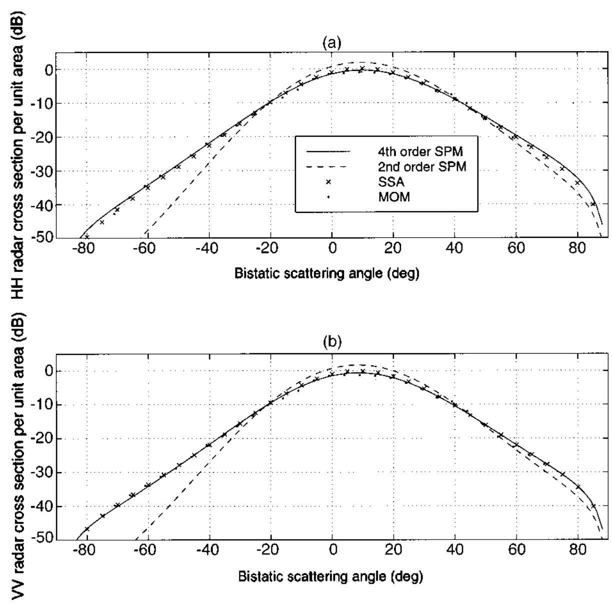

Fig. 2. Comparison of SPM, SSA, and MOM bistatic scattering cross sections per unit area for a Gaussian correlation function surface with $h = 0.06\lambda$ and $l = \lambda$, $\theta_i = 10°$, and $\epsilon = 3$. (a) *HH*, (b) *VV*.

quadrature. In-plane, horizontally polarized (HH) bistatic scattering cross sections at second and fourth order are compared in Fig. 1 for $h = 0.06\lambda$, $l = \lambda$, and $\epsilon = 3$ and with a horizontally polarized plane wave incident at 10° from normal incidence. The definition of θ_s in this figure results in forward scattering occurring at $\theta_s = 10°$, while backscattering occurs at $\theta_s = -10°$. Curves for the $\zeta^{(2)}\zeta^{(2)*}$ and $\zeta^{(3)}\zeta^{(1)*}$ contributions to fourth-order results are also included and illustrate that both terms contribute to the total results. The $\zeta^{(3)}\zeta^{(1)*}$ terms reduce second-order cross sections primarily at near-specular angles, whereas the $\zeta^{(2)}\zeta^{(2)*}$ terms are more important at nonspecular angles. Second- and fourth-order *HH* and *VV* SPM predictions are compared with the zeroth-order SSA in Fig. 2, and fourth-order SPM results are observed to be in good agreement with the SSA. Results from a 50-realization Monte Carlo MOM simulation are also included in Fig. 2 and confirm the accuracy of the SSA and the fourth-order SPM for this case. Monte Carlo results were calculated with the canonical-grid technique[19] in a four-scalar-function-unknown MOM for a penetrable surface[20] to improve computational efficiency and were obtained for surfaces size $16\lambda \times 16\lambda$ sampled in 128×128 points. The tapered incident field described in Ref. 19 with $g = 5$ was used to eliminate edge scattering effects, but it causes inaccuracies for large bistatic scattering angles, so MOM results are included only for scattering angles within 70° of normal in Fig. 2. Further comparisons of the SSA and the fourth-order SPM were performed for several different h and l values, and fourth-order predictions were found to provide improved agreement with the SSA for surfaces with small slopes, $h \leqslant 0.06\lambda$, and $l \leqslant \lambda$.

To demonstrate a surface for which there is a third-order power contribution, a periodic asymmetric pyramidal surface is considered next, as shown in Fig. 3. This surface is essentially a square-based pyramid, but the peak of the pyramid is shifted to an off-center position along the y axis. The surface is defined to have zero mean, and in addition it is passed through a low-pass filter that removes all Fourier coefficients with $(n^2 + m^2)^{1/2} > 8$. This low-pass filter is used to avoid the slope discontinuity that occurs at the peak of the ideal pyramid. Reflection from a grating with $P_x = P_y = 8\lambda$, pyramid peak-to-peak amplitude $A = 0.2\lambda$, peak location at $y = 6.5\lambda$, and $\epsilon = 10 + i10$ is considered for a horizontally polarized plane wave incident at 40° from normal incidence and at azimuthal angle 25° from the x axis. Percent reflectivities in *HH* and *VH* polarizations from the SPM at second, third, and fourth orders are listed in Tables 1 and 2, respectively for near-specular modes with n' and m' indices ranging from -1 to 1. Note that the flat-surface reflectivity $|\Gamma_h|^2$ has been removed from the $n' = m' = 0$ mode of the *HH* results. Numerical EBC results are also included in the table, and the percent difference from SPM predictions at each order are provided. Since it can be shown that the bispectrum of this surface does not vanish, a third-order correction is present in

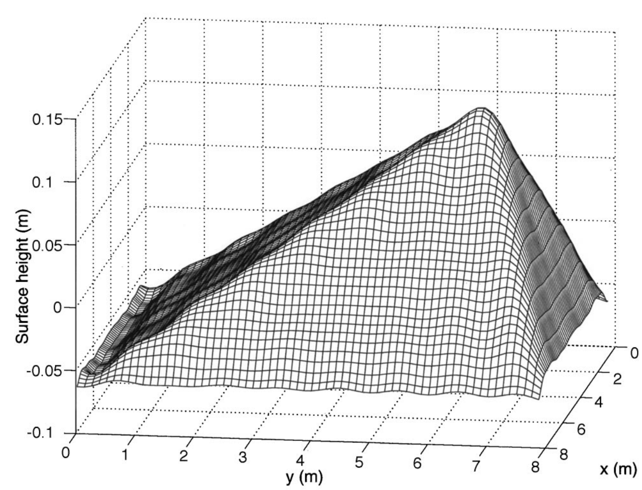

Fig. 3. Asymmetric pyramidal surface.

Table 1. *HH* Percent Reflectivities in Near-Specular Modes for an Asymmetric Pyramidal Grating

n'	m'	2nd Order	% Error	3rd Order	% Error	4th Order	% Error	EBC
−1	−1	0.4179	12.1026	0.5313	11.7512	0.4864	2.3203	0.4754
−1	0	1.8678	16.4719	1.8687	16.5297	1.5803	1.4544	1.6036
−1	1	0.3703	47.0581	0.2765	9.8100	0.2407	4.4029	0.2518
0	−1	1.5182	14.1400	1.9825	12.1124	1.8150	2.6402	1.7683
0	0	−9.5415	9.5528	−9.5390	9.5236	−9.0517	3.9289	−8.7095
0	1	1.4223	53.4238	1.0175	9.7531	0.8786	5.2270	0.9271
1	−1	0.3476	12.0005	0.4306	9.0083	0.4020	1.7676	0.3950
1	0	1.5900	12.0881	1.5890	12.0178	1.4032	1.0782	1.4185
1	1	0.3246	35.3334	0.2564	6.9021	0.2336	2.5837	0.2398

Table 2. *VH* Percent Reflectivities in Near-Specular Modes for an Asymmetric Pyramidal Grating

n'	m'	2nd Order	% Error	3rd Order	% Error	4th Order	% Error	EBC
−1	−1	0.0077	9.5123	0.0095	11.5730	0.0087	1.7186	0.0085
−1	0	0.0223	14.1052	0.0226	15.4892	0.0195	0.0952	0.0195
−1	1	0.0382	50.9989	0.0275	8.8139	0.0239	5.4594	0.0253
0	−1	0.0710	11.9212	0.0900	11.7378	0.0825	2.3427	0.0806
0	0	0.0000	100.0000	−0.0000	100.0003	0.0000	1.1943	0.0000
0	1	0.0542	65.0593	0.0360	9.6698	0.0304	7.3185	0.0328
1	−1	0.0288	9.4042	0.0347	9.4267	0.0322	1.5075	0.0317
1	0	0.0123	16.0286	0.0122	15.1979	0.0106	0.5083	0.0106
1	1	0.0032	54.8619	0.0022	5.7218	0.0020	4.3421	0.0021

these tables. Third-order power contributions are composed primarily of the $2\,\mathrm{Re}\{\alpha_{nm}^{(1)*}\alpha_{nm}^{(2)}\}$ terms, although a small correction to the specular-reflection coefficient is obtained from the third-order field expressions. A similar level of corrections is obtained at fourth order in these tables, although the fourth-order specular term is not included for reasons previously discussed. Again, the comparison with EBC results is not exact for this moderately rough surface, but it shows the improvement obtained as higher-order SPM terms are included.

9. CONCLUSIONS

A systematic procedure for determining higher-order terms in the small-perturbation method has been presented and has been applied to determine SPM scattered fields up to third order in surface height. Although the procedure is based on the Rayleigh hypothesis for deterministic, periodic surfaces, the results can be generalized to the nonperiodic and stochastic surface cases as well. Sample results that illustrated the utility of these new terms were presented; in particular it was shown that third-order field terms can contribute to scattered powers at fourth order even for surfaces with vanishing bispectra and to a specular-reflection coefficient correction at third order if the surface has a nonvanishing bispectrum, implying a horizontal or vertical skewness in the surface profile. The final case is of particular interest for passive remote sensing of the ocean, since it is the horizontal skewness of the ocean surface that gives rise to first azimuthal harmonic variations of brightness temperatures. In addition, the fact that the SPM produces a series in surface slope, not height, when applied to the calculation of thermal emission makes these higher-order terms very applicable to the emission problem. Such applications will be considered in future work.

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