Ordinal Game Theory and Applications – A New Framework For Games Without Payoff Functions

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ABSTRACT

Decision-making problems in engineering, business, management, and economics that involve two or more decision-makers with competing objectives are often optimized using the theory of games. This theory, initially developed by Von Neumann and Morgenstern, and later by Nash, requires that each point in the decision space be mapped, through a payoff function, into a real number representing the value of the collective set of decisions to each decision maker. This theory, which is cardinal in nature, requires that each decision-maker determine its decision by maximizing its payoff function taking into account the choice of decisions by all other decision-makers. While this theory has been very useful in addressing some aspects of quantitative decision-making in engineering and economics, it has not been able to adequately address qualitative problems in fields such as social and political sciences, as well as a large segment of complex problems in engineering, business and management imbedded in a competitive environment. The main reason for this is the inherent difficulty in defining an adequate payoff function for each decision maker in these types of problems.

In this paper, we present a theory where, instead of a payoff function, the decision-makers are able to rank order their decisions against decision choices by the other decision-makers. Such a rank ordering could be the result of personal, subjective, preferences derived from qualitative analysis, as is the case in many social sciences problems. In such problems a heuristic, knowledge-based, rank ordering of decision choices in a finite decision space can be viewed as a first step in the process of modeling complex problems for which a mathematical description is usually extremely difficult, if not impossible, to obtain. In order to distinguish between these two types of games, we will refer to traditional payoff-based games as "Cardinal Games" and to these new types of rank ordering-based games as "Ordinal Games". In this paper, we review the theory of ordinal games and discuss associated solution concepts such as the Nash equilibrium. We will also show that these solutions are general in nature and can be characterized, in terms of existence and uniqueness, with conditions that are more intuitive and much less restrictive than those of the traditional cardinal games. We will illustrate these concepts with several examples of deterministic matrix games, including an example of team composition and task assignment by a top commander in a military operation where payoff functions are not readily available. We feel that this new theory of ordinal games will be very useful when dealing with complex decision-making problems that involve more than one decision makers in a competitive environment.

1. Introduction

Systems that involve more than one decision-maker in competition are often optimized using the theory of games. This theory, initially developed by Von Neumann and Morgenstern [1], and later by Nash [2], requires that the collective decisions of the decision-makers be mapped through payoff functions[1] into a set of real numbers representing the values of these decisions to each of the decision-makers. In order to illustrate this concept, let us consider the simple matrix game illustrated in Figure 1.

		(DM2)		
		y_1	y_2	y_3
(DM1)	x_1	\$5,100 \$3,200	\$8,000 \$3,200	\$5,100 \$2,900
	x_2	\$6,000 \$9,000	\$5,100 \$8,300	\$1,900 \$6,300
	x_3	\$8,000 \$7,300	\$4,200 \$4,400	\$1,100 \$7,300

Figure 1: A Simple Matrix Game

DM1 (Decision-Maker 1) must choose from three options in his decision space $X = \{x_1, x_2, x_3\}$ and DM2 has also three options $Y = \{y_1, y_2, y_3\}$ to choose from. For each pair of choices $\{x_i, y_j\}$ the entries $P_1(x_i, y_j)$ and $P_2(x_i, y_j)$ in the matrix of Figure 1 represent costs associated with these choices that each DM will incur. The objective of DM1 is to choose an option $x_i \in X$ to minimize his costs C_1, and the objective of DM2 is to choose an option $y_j \in Y$ so as to minimize his costs C_2. Obviously, in this example, both decision-makers are interdependent in that the costs incurred by each is determined not only by its own choice, but also by the choice made by the other DM. Hence, in making a choice each DM has to take into account the choice made by the other DM.

[1] Other common terms used instead of payoff function include utility function, objective function, cost function, loss function, performance function, profit function, etc. A best selection of decision will involve either maximizing or minimizing this function depending on its definition.

Clearly, the usefulness of game theory as in the example described above is limited to problems where the payoff functions (costs) can be expressed, and mathematically defined, for all decision-makers in the game. It is not difficult, however, to imagine many real life decision-making problems where the payoff functions cannot be easily determined. For example, problems that most individuals face in buying a house, planning a vacation, or resolving a conflict, cannot be easily expressed in terms of a payoff function. Even, in such fields as military decision-making applications where the theory of games has traditionally received considerable attention, planning and managing an air operation in the presence of an intelligent adversary could be extremely difficult, if not impossible, to formulate using this theory of games. In these problems, the main difficulty lies in the inability to formulate appropriate payoff functions for all decision-makers involved. On the other hand, in the majority of such problems, instead of payoff functions, the decision-makers may have certain preferences that can be expressed easily as a rank ordering of the various options available to them. Furthermore, these preferences may be completely subjective in nature reflecting certain biases or experiences that they may have. A new theory of games that deals with rank ordering of the options, rather than payoff functions as a basis for optimization has recently been developed by the authors [3]. For the sake of completeness, we will review below the aspects of this theory that are relevant to business decision making.

As a simple example, to illustrate the main idea of rank ordering in games, consider the situation of two friends, John and Mary, planning a weekend vacation together [3]. John likes to go the mountains and Mary prefers the beach. They both like to be together, if possible, but they have strong preferences for the various options available to them. Let us assume that of the nine possible options, John ranks his preferences as illustrated in the table (or matrix) of Figure 2.

Mary may have completely different preferences. Let us suppose that hers are ranked as shown in

Figure 3. Superimposing both preferences[2] we get the matrix game shown in Figure 4.

		Mary (P2)		
		Mountain	Beach	Home
John (P1)	Mountain	1	6	5
	Beach	8	4	9
	Home	7	2	3

Figure 2: John's Preferences

		Mary (P2)		
		Mountain	Beach	Home
John (P1)	Mountain	7	8	9
	Beach	2	1	3
	Home	4	6	5

Figure 3: Mary's Preferences

Note that in contrast to the game of Figure 1, the game of Figure 4 is formulated using preferential rankings instead of payoff functions. The entries in this game are positive integers representing the order (or ranking) of the various options for each player. In order to differentiate such games from the traditional, cardinal games, we will refer to them as Ordinal Games [3]. Note that in the ordinal game of Figure 4, all nine options are rank ordered for each player. For example, both going to the mountains is John's 1st preference but Mary's 7th preference, and both going to the beach is Mary's 1st preference but John's 4th preference; however, John staying at home while Mary going to the beach is his 2nd preference but her 6th and so on. These are purely subjective preferences that may have no associated payoff or value.

		Mary (P2)		
		Mountain	Beach	Home
John (P1)	Mountain	1 , 7	6 , 8	5 , 9
	Beach	8 , 2	4 , 1	9 , 3
	Home	7 , 4	2 , 6	3 , 5

Figure 4: John and Mary's Ordinal Game

In many ways, *ordinal games* can be viewed as extensions of *ordinal optimization problems* in the same manner as *cardinal games* are extensions of *cardinal optimization problems*. Ordinal optimization is a means of finding good, better, or best designs from a set of ordered options rather than using a formal cardinal process of calculating the payoff cost or value of each option. It is a simple and yet very effective method of optimizing systems as demonstrated by Ho *et al* [4] in optimizing Discrete Event Dynamic Systems (DEDS).

The use of ordinal methods instead of payoff functions in decision-making problems is a concept that has received considerable attention in the past 20 years or so. The Analytic Hierarchy Process (AHP) developed by Saaty in 1980 [5-6] is a very effective tool for optimizing complex multi-criteria decision-making problems. The AHP requires that the decision-maker consider judgments, possibly subjective, about the relative importance of each criterion and specify a preference for each decision option with respect to each criterion. The outcome of this process is a prioritized ranking, indicating an overall

[2] In this table, we use the number 1 to indicate 1st preference, 2 for 2nd preference and so on. Since there are 9 options, the number 9 will correspond to the 9th (last) preference. In the case of two or more options being equally ranked, the last preference will be less than 9th. Note that the use of integer numbers to indicate the ordering of the preferences, as we do in this paper, is purely optional. Symbols such as a, b, c, etc. or *, #, @, etc. could have been used as well.

preference, of the various decision options available to the decision maker. Along the same lines, the Theory of Moves developed by Brams in 1994 [7] deals with models for conflict resolution that involve successive unilateral actions by the decision makers. These models, which differ in several crucial ways from game models [8], are treated using an ordinal methodology that avoids the use of utility functions. Other methods that rely on an ordinal approach for representing preferences for the decision makers include the Graph Model Conflict Resolution (GMCR) developed by Kilgour *et al* [9-11] and the Drama Theory developed by Howard [12]. This ordinal approach has recently been introduced [3] as an alternative to the traditional formulation of zero and nonzero sum games as described by Von Neumann and Morgenstern [1], and Nash [2]. The theory of ordinal games is conceptually simple in that it can handle complex real world problems in many business and management decision-making problems and yet it is mathematically robust so that its results are meaningful and consistent with those of its counterpart, the theory of cardinal games.

2. Cardinal to Ordinal Games

An interesting aspect of the theory of ordinal games is that it can also handle the traditional cardinal games as well. This is so because cardinal games can be easily transformed into ordinal games as we illustrate in this simple example. Let us consider again the cardinal game example of Figure 1. Clearly, it is possible to map the costs for each DM into a ranking of preferences. For DM1, the ranking of the options would be as shown in Figure 5, and for DM2 as in Figure 6. Using these ranking instead of the payoffs, the matrix for the game can be reformulated as shown in Figure 7. The entries in this new matrix are now the rankings of each pair of choices for DM1 and DM2 respectively. Thus, for example, the pair $\{x_1, y_1\}$ is ranked 4th best for F=DM1 and 2nd best for DM2 and so on. The highest ranked option for DM1 is clearly $\{x_3, y_3\}$ and for DM 2 is $\{x_1, y_3\}$.

Costs	Rank	
$1,100	1	Best
$1,900	2	↓
$4,200	3	↓
$5,100	4	↓
$6,000	5	↓
$8,000	6	Worst

Figure 5: Rank Ordering for DM 1

Costs	Rank	
$2,900	1	Best
$3,200	2	↓
$4,400	3	↓
$6,300	4	↓
$7,300	5	↓
$8,300	6	↓
$9,000	7	Worst

Figure 6: Rank Ordering for DM2.

		DM2		
		y_1	y_2	y_3
DM1	x_1	4 , 2	6 , 2	4 , 1
	x_2	5 , 7	4 , 6	2 , 4
	x_3	6 , 5	3 , 3	1 , 5

Figure 7: Ordinal Game equivalent of the Cardinal Game of Figure 1.

3. Ordinal Matrix Games: Formulation [3]

Let us consider a game with two decision makers DM1 and DM2. Let $X = \{x_1, x_2, ..., x_n\}$ represent the decision space for DM1 and $Y = \{y_1, y_2, ..., y_m\}$ represent the decision space for DM2. Ordinal games are based on the concept of preferential rank ordering. Let $R_1(x,y)$ and $R_2(x,y)$ for $x \in X$ and $y \in Y$ denote the $n \times m$ matrices of rank ordering of the decision pair $\{x,y\} \in X \times Y$ for DM1 and DM2 respectively. We note that the entries of $R_1(x,y)$ and $R_2(x,y)$ are integers

$$R_1(x_i,y_j) = \{1,2,.....,W_1\} \quad \text{for } i=1,...,n \text{ and } j=1,...,m \tag{1}$$

and

$$R_2(x_i,y_j) = \{1,2,.....,W_2\} \quad \text{for } i=1,...,n \text{ and } j=1,...,m \tag{2}$$

where W_1 and W_2 represent the last ranked (worst) options for DM1 and DM2 respectively. Note that W_1 and W_2 are bounded above by the product $n \times m$, allowing for some decision pairs to have duplicate (same) rank. Thus, the notation $R_i(x_a,y_b) = k$ means that the decision pair $\{x_a,y_b\}$ is ranked as the k^{th} best option for DMi. It is possible for two decision pairs to be equally ranked. This would be indicated by $R_i(x_a,y_b) = R_i(x_c,y_d)$ and would imply that for DMi, the decision pair $\{x_a,y_b\}$ has the same preference (rank) as $\{x_c,y_d\}$.

Definition 1. A two-decision-maker matrix ordinal game is defined as a pair of $n \times m$ matrices R_1 and R_2. The ij^{th} entries of R_1 and R_2 represent the preferential rankings of the decision pair $\{x_i,y_j\}$ for DM1 and DM2 respectively.

As an example, for the ordinal game of Figure 4, we have:

$$R_1 = \begin{bmatrix} 1 & 6 & 5 \\ 8 & 4 & 9 \\ 7 & 2 & 3 \end{bmatrix} \quad \text{and} \quad R_2 = \begin{bmatrix} 7 & 8 & 9 \\ 2 & 1 & 3 \\ 4 & 6 & 5 \end{bmatrix} \tag{3}$$

Definition 2. Let u be an $n \times 1$ (column or row) vector whose entries are real numbers. We will define *an associated rank ordered vector* u^o as the $n \times 1$ vector in which each number is replaced by its order (rank) in the set $\{u_i \text{ for } i=1,...,n\}$ starting with the smallest.

Note that in the above definition, it is implied that entries in u that are equal are assigned the same rank. As an example, if $u = [7.2,\ 26.2,\ 4.5,\ 8.6]$ then $u^o = [2,\ 4,\ 1,\ 3]$, and if $u = \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$ then $u^o = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$.

4. Generalized Nash Solutions for Ordinal Games

The Nash solution for cardinal games [2] represents an equilibrium point when each decision-maker reacts to the other decision-maker by choosing the option that gives him the best payoff. In order to be able to define a concept similar to the Nash solution for ordinal games, we first need the following definition.

Definition 3. Let M be an $n \times m$ matrix whose entries are real numbers. We will define a *column rank ordered matrix* M^{co} as the corresponding $n \times m$ matrix in which each column vector is replaced by its corresponding ordered column vector. That is,

$$M^{co} = \left[m^o_{c1} \vdots\ m^o_{c2} \vdots\ \cdots\ \vdots m^o_{cm} \right]$$

Similarly, we define a *row rank ordered matrix* M^{ro} as the corresponding $n \times m$ matrices in which each row vector is replaced by its corresponding ordered row vector. That is,

$$M^{ro} = \begin{bmatrix} m^o_{r1} \\ \cdots \\ m^o_{r2} \\ \cdots \\ \vdots \\ \cdots \\ m^o_{rn} \end{bmatrix}$$

As an illustrative example, if $M = \begin{bmatrix} 4 & 7 & 1 \\ 5 & 3 & 4 \\ 6 & 2 & 6 \end{bmatrix}$,

then $M^{co} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & 2 & 2 \\ 3 & 1 & 3 \end{bmatrix}$ and $M^{ro} = \begin{bmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \\ 2 & 1 & 2 \end{bmatrix}$

Definition 4. Given an ordinal game defined by the matrices R_1 and R_2, each pair of decisions $\{x_i, y_j\}$ for $i = 1,...,n$ and $j = 1,...,m$ is defined as a **Generalized Nash** (GN) solution of order $\{R_1^{co}(x_i, y_j),\ R_2^{ro}(x_i, y_j)\}$.

As an example, for the ordinal game of Figure 4, for which R_1 and R_2 are given in (3), we have

$$R_1^{co} = \begin{bmatrix} 1 & 3 & 2 \\ 3 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix} \text{ and } R_2^{ro} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{bmatrix}. \qquad (4)$$

Hence, $\{x_1, y_1\}$ is a GN solution of order {1,1} and $\{x_1, y_2\}$ is a GN solution of order {3,2}, etc. A GN solution of order {3,2} is a Nash equilibrium point when DM1 reacts to choices of DM2 by choosing its 3rd ranked option while DM2 reacts to choices of DM1 by choosing its 2nd ranked option. Similarly, if both decision-makers always react by choosing their 2nd preference choice, then the GN equilibrium will be of order {2,2}. A GN of order {2,2} does not exist in the above example. On the other hand, a GN solution of order {3,3} does exist. It corresponds to $\{x_2, y_3\}$ and represents a GN equilibrium point when each player reacts by choosing its 3rd (i.e. worst) option. From a practical point of view, this would be the <u>least desirable</u> (or worst ranked) Nash equilibrium for the game.

The above definition essentially says that in an ordinal game every pair of choices is a GN solution of a certain order. A Generalized Nash solution of order $\{R_1^{co}(x_i, y_j),\ R_2^{ro}(x_i, y_j)\}$ represents a Nash equilibrium point when DM1 reacts by choosing its $R_1^{co}(x_i, y_j)^{th}$ preference option and DM2 reacts by choosing its $R_2^{ro}(x_i, y_j)^{th}$ preference option. Clearly, this concept of Generalized Nash solutions for ordinal games is much richer than the corresponding concept in cardinal games.

Definition 5. For an ordinal game if there exists a Generalized Nash (GN) solution of order {1,1}, then we shall call this Nash solution an **Optimal Nash** (ON) solution for the game.

As an example, for the ordinal game $\{R_1, R_2\}$ of Figure 4, for which $\{R_1^{co}, R_2^{ro}\}$ are shown in (4) above, the pair $\{x_1, y_1\}$ is an Optimal Nash solution for the game. This corresponds to the <u>most desirable</u> (or highest ranked) Nash equilibrium for the game. Similarly, for the ordinal game of Figure 7, we have

$$R_1^{co} = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 2 & 2 \\ 3 & 1 & 1 \end{bmatrix}, \quad R_2^{ro} = \begin{bmatrix} 2 & 2 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix},$$

and the Optimal Nash solution is $\{x_3, y_2\}$

The concept of an Optimal Nash solution for ordinal games is equivalent to the standard concept of Nash solution in cardinal games. However, as is well known, Nash solutions in pure strategies do not always exist in cardinal games. In ordinal games, the concept of Generalized Nash solutions provides for numerous alternative Nash solutions in case the Optimal Nash solution does not exist.

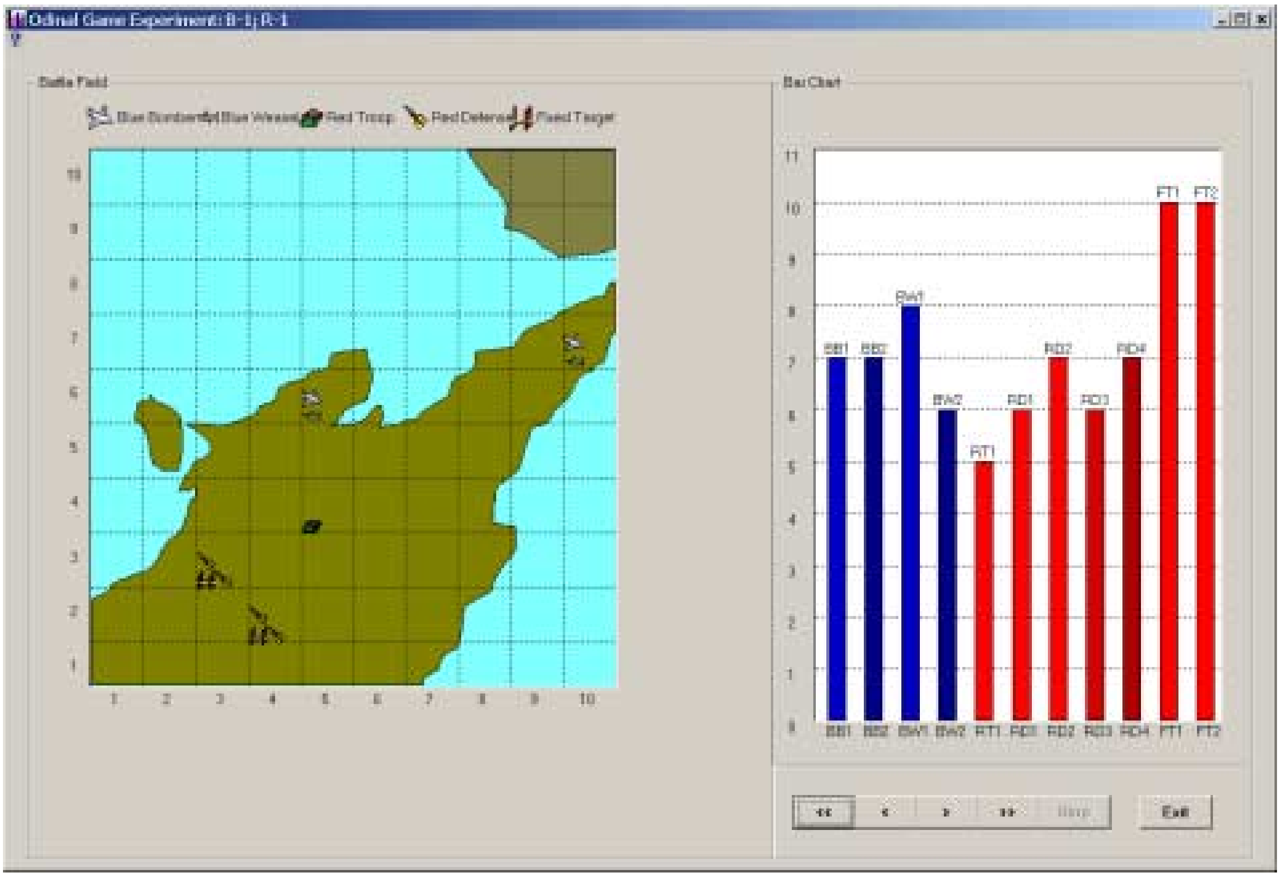

Figure 8: A Military Example

5. A Military Application of an Ordinal Game

Consider a military scenario of a battle between two opposing forces as illustrated in Figure 8. One force, which we will refer to as the Blue force, consists of Bombers (BBs) and fighter planes, (BWs), also referred to as weasels. This force is given the mission of destroying two fixed targets (for example an airport and a bridge) that are defended by an enemy force, which we will refer to as the Red force. The Red force consists of ground troops (RTs) and defense units RDs (surface to air missile SAM batteries) placed in the vicinity of the fixed targets. A game theoretic attrition type model that can be used for optimal decision-making associated with the command and control of these forces has been developed [13, 14]. In this paper, we will use this model to illustrate how ordinal game theory can be used in the decision-making process at the top commanders level of both the Blue and Red forces. An issue that each top commanders face is how to best team their forces and assign the various tasks to be performed in order to best achieve the desired results. Clearly, this team composition and task assignment decision constitute a game in itself since what one top commander chooses to do is very much influenced by what the other does. Furthermore, since decisions at the top commander level can be subjective in nature, based on the specific preferences of the commander, this game is best formulated, using ordinal game theory.

As can be seen in Figure 8, in this example, the Blue force consists of two groups of Blue bombers, BB1 and BB2, and two groups of Blue Fighters, BW1 and BW2. The Red force includes the two adjacent fixed targets, FT1 and FT2, defended by four groups of Red defense units RD1, RD2, RD3, and RD4 and one group of Red troop RT1. We will assume that the battle will continue for as long as needed until the goal of the Blue force is accomplished or until the Blue units spend all available weapons before accomplishing their tasks. The description and initial equipment for each unit are shown in the bar chart in Figure 8 and listed in Table 1.

	Type	# of entities	Average Weapons	Max. Salvo
BB1	F4 bombers	7	4	1
BB2	F4 bombers	7	4	1
BW1	F2-E fighters	8	4	1
BW2	F2-E fighters	6	3	1
RT1	Armored vehicles	50	3	0.5
RD1	Fixed SAMs & Radar	6	15/6	5/6
RD2	Fixed SAMs & Radar	7	18/7	6/7
RD3	Fixed SAMs & Radar	6	15/6	5/6
RD4	Fixed SAMs & Radar	7	18/7	6/7
FT1	Airport	10	N/A	N/A
FT2	Bridge	10	N/A	N/A

Table 1. Description and Initial Equipment of Units in the Example

The top commander of each force can team various units in order to best achieve the forces' objectives. Let us assume that each commander has the following three options of team compositions and task assignments (normally, many more options could be considered. In this example, due to space limitations, we will consider only three for each commander):

<u>The Offensive Options for Blue Commander</u>:

<u>Option X</u>: BB1+BW1 assigned to FT1, and BB2+BW2 assigned to FT2.
<u>Option Y</u>: BB2+BW2 assigned to FT1, and BB1+BW1 assigned to FT2.
<u>Option Z</u>: BB1+BW1+BB2+BW2 assigned to FT1 first, then to FT2 after destroying FT1.

<u>The Defensive Options for Red Commander</u>:

<u>Option A</u>: RD1+RD2 assigned to defend FT1, and RD3+RD4 to defend FT2.
<u>Option B</u>: RD1+RD3 assigned to defend FT1, and RD2+RD4 to defend FT2.
<u>Option C</u>: RD1+RD2+RD3 assigned to defend FT1, and RD4 to defend FT2.

Each of the above nine possible combinations of options to conduct the battle was modeled and its corresponding Nash solution obtained using the results of [13, 14]. For each pair of options the remaining forces at the end of the battle are illustrated in a bimatrix form in Figure 9. Each entry in this figure shows pictorially, in a bar chart style, the remaining units on the Blue side and the remaining units on the Red side for the given pair of choices of team composition and assignments by the top commanders. Clearly, at this point, the decision as to which option is preferable than the others becomes subjective, depending on the particular goals of the top commander. Let us assume that a subjective rank ordering of these options from both top commanders' perspectives is as shown in Figure 10. It is important to point out that this rank ordering could be easily influenced by the top commanders past experiences, and could vary substantially from one commander to another.

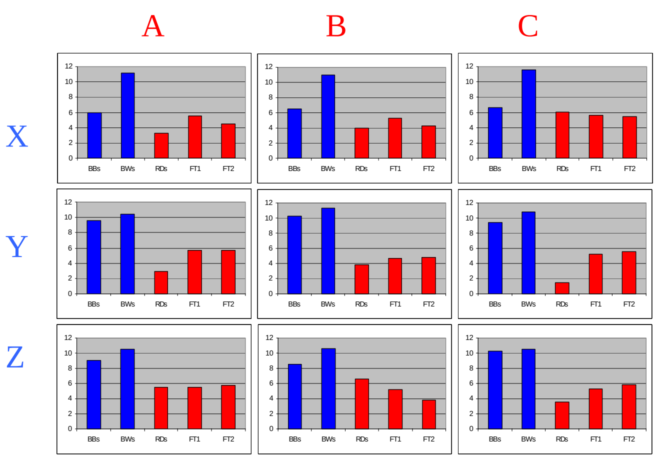

Figure 9: Outcome of battle for all nine possible combinations of options.

		Red Commander Options		
		A	B	C
Blue Commander Options	X	9, 3	7, 2	8, 1
	Y	5, 6	1, 7	3, 9
	Z	6, 5	4, 4	2, 8

Figure 10: A Ranking of the Options for Each Top Commander

In order to derive the Nash equilibrium for this problem, the rank ordered matrices $\{R_1^{co}(x_i, y_j),\ R_2^{ro}(x_i, y_j)\}$ as defined in definition 4 will be as follows:

$$R_1^{co} = \begin{bmatrix} 3 & 3 & 3 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \quad \text{and} \quad R_2^{ro} = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}$$

Clearly, the Optimal Nash solution for this problem is {Y, A} resulting in the sixth preference for the Blue commander and 5th preference for the Red commander.

6. Conclusion

In this paper, we reviewed the main results of a new theory of games where, instead of calculating payoffs, the decision-makers are able to rank order their decision choices against decision choices of the other decision-makers. We labeled these types of games as *Ordinal Games*. We developed the concepts of generalized Nash solutions for ordinal games and showed that these are much richer concepts than their counterparts in traditional cardinal games. We also presented several examples to illustrate the usefulness of this theory. We feel that this new theory of ordinal games will provide a very useful alternative in decision-making problems that cannot be formulated using the traditional payoff function approach.

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