Letters

Stable Multi-Input Multi-Output Adaptive Fuzzy/Neural Control

Raúl Ordóñez and Kevin M. Passino

***Abstract*— In this letter, stable direct and indirect adaptive controllers are presented that use Takagi–Sugeno (T–S) fuzzy systems, conventional fuzzy systems, or a class of neural networks to provide asymptotic tracking of a reference signal vector for a class of continuous time multi-input multi-output (MIMO) square nonlinear plants with poorly understood dynamics. The direct adaptive scheme allows for the inclusion of *a priori* knowledge about the control input in terms of exact mathematical equations or linguistics, while the indirect adaptive controller permits the explicit use of equations to represent portions of the plant dynamics. We prove that with or without such knowledge the adaptive schemes can "learn" how to control the plant, provide for bounded internal signals, and achieve asymptotically stable tracking of the reference inputs. We do not impose any initialization conditions on the controllers and guarantee convergence of the tracking error to zero.**

***Index Terms*— Direct adaptive control, fuzzy control, indirect adaptive control, MIMO nonlinear systems, neural control.**

I. Introduction

FUZZY systems and neural networks-based control methodologies have emerged in recent years as a promising way to approach nonlinear control problems. Fuzzy control, in particular, has had an impact in the control community because of the simple approach it provides to use heuristic control knowledge for nonlinear control problems. However, in the more complicated situations, where the plant parameters are subject to perturbations or when the dynamics of the system are too complex to be characterized reliably by an explicit mathematical model, adaptive schemes have been introduced that gather data from on-line operation and use adaptation heuristics to automatically determine the parameters of the controller. See, for example, the techniques in [1]–[7]; to date, no stability conditions have been provided for these approaches. Recently, several stable adaptive fuzzy control schemes have been introduced [8]–[12]. Moreover, closely related neural control approaches have been studied [13]–[18].

In the above techniques, emphasis is placed on control of single-input single-output (SISO) plants (except for [4], which can be readily applied to multi-input multi-output (MIMO) plants as it is done in [5] and [6], but lacks a stability analysis). In [19], adaptive control of MIMO systems using multilayer neural networks is studied. The authors consider feedback linearizable, continuous time systems with general relative degree, and utilization of neural networks to develop an indirect adaptive scheme. These results are further studied and summarized in [20]. The scheme in [19] requires the assumptions that the tracking and neural network parameter errors are initially bounded and sufficiently small and they provide convergence results for the tracking errors to fixed neighborhoods of the origin.

In this paper, we present direct [21] and indirect [22] adaptive controllers for MIMO plants with poorly understood dynamics or plants subjected to parameter disturbances, which extend the results in [8]. We use Takagi–Sugeno (T–S) fuzzy systems or a class of neural networks with two hidden layers as the basis of our control schemes. We consider a general class of square MIMO systems decouplable via static nonlinear state feedback and obtain asymptotic convergence of the tracking errors to zero (rather than to a bounded neighborhood of the origin) and boundedness of the parameter errors as well as state boundedness provided the zero dynamics of the plant being exponentially attractive. The stability results do not depend on any initialization conditions and we allow for the inclusion in the control algorithm of *a priori* heuristic or mathematical knowledge about what the control input should be in the direct case or about the plant dynamics in the indirect case. Note that while the indirect approach is a fairly simple extension of the corresponding SISO case in [8] (the results in [19] and [20] are also on indirect adaptive control), the direct adaptive case is not. The direct adaptive method turns out to require more restrictive assumptions than the indirect case, but is perhaps of more interest because as far as we are aware, no other direct adaptive methodology with stability proof for the class of MIMO systems we consider here has been presented in the literature. The results in this paper are nonlocal in the sense that they are global whenever the change of coordinates involved in the feedback linearization of the MIMO system is global.

The paper is organized as follows. In Section II, we introduce the MIMO direct adaptive controller and give a proof of the stability results. In Section III, we outline the MIMO indirect adaptive controller giving just a short sketch of the proof (the complete proof can be found in [22]). In Section IV, we present simulation results of the direct adaptive method applied first to a nonlinear differential equation that satisfies all controller assumptions as an illustration of the method and then to a two-link robot. The robot is an interesting practical application and it is of special interest here because it does *not* satisfy all assumptions of the controller; however, we show

Manuscript received July 26, 1997; revised January 1, 1999. This work was supported in part by the National Science Foundation under Grant EEC 9315257.

The authors are with the Department of Electrical Engineering, Ohio State University, Columbus, OH 43210 USA.

Publisher Item Identifier S 1063-6706(99)04936-X.

how the method can be made to work in spite of this fact. In Section V, we provide the concluding remarks.

II. Direct Adaptive Control

Consider the MIMO square nonlinear plant (i.e., a plant with as many inputs as outputs [23], [24]) given by

$$\begin{aligned}\dot{\mathbf{X}} &= f(\mathbf{X}) + g_1(\mathbf{X})u_1 + \cdots + g_p(\mathbf{X})u_p\\ y_1 &= h_1(\mathbf{X})\\ &\vdots\\ y_p &= h_p(\mathbf{X})\end{aligned} \tag{1}$$

where $\mathbf{X} = [x_1, \cdots, x_n]^\top \in \Re^n$ is the state vector, $\mathbf{U} := [u_1, \cdots, u_p]^\top \in \Re^p$ is the control input vector, $\mathbf{Y} := [y_1, \cdots, y_p]^\top \in \Re^p$ is the output vector, and f, g_i, h_i, $i = 1, \cdots, p$ are smooth functions. If the system is feedback linearizable [24] by static state feedback and has a well-defined vector relative degree $\mathbf{r} := [r_1, \cdots, r_p]^\top$, where the r_i's are the smallest integers such that at least one of the inputs appears in $y_i^{(r_i)}$, the input–output (IO) differential equations of the system are given by

$$y_i^{(r_i)} = L_f^{r_i} h_i + \sum_{j=1}^{p} L_{g_j}(L_f^{r_i-1} h_i)u_j \tag{2}$$

with at least one of the $L_{g_j}(L_f^{r_i-1}h_i) \neq 0$ [note that $L_f h(\mathbf{X})$: $\Re^n \to \Re$ is the Lie derivative of h with respect to f, given by $L_f h(\mathbf{X}) = (\partial h)/(\partial \mathbf{X}) f(\mathbf{X})$]. Define (for convenience) $\alpha_i(\mathbf{X}) := L_f^{r_i} h_i$ and $\beta_{ij}(\mathbf{X}) := L_{g_j}(Lf^{r_i-1}h_i)$. In this way, we may rewrite the plant's IO equation as

$$\underbrace{\begin{bmatrix} y_1^{(r_1)}\\ \vdots \\ y_p^{(r_p)}\end{bmatrix}}_{\mathbf{Y}^{(\mathbf{r})}(t)} = \underbrace{\begin{bmatrix}\alpha_1\\ \vdots\\ \alpha_p\end{bmatrix}}_{\mathbf{A}(\mathbf{X},t)} + \underbrace{\begin{bmatrix}\beta_{11} & \cdots & \beta_{1p}\\ \vdots & \ddots & \vdots\\ \beta_{p1} & \cdots & \beta_{pp}\end{bmatrix}}_{\mathbf{B}(\mathbf{X},t)} \underbrace{\begin{bmatrix}u_1\\ \vdots\\ u_p\end{bmatrix}}_{\mathbf{U}(t)}. \tag{3}$$

Consider the ideal feedback linearizing control law $\mathbf{U}^* = [u_1^*, \cdots, u_p^*]^\top$ given by

$$\mathbf{U}^* = \mathbf{B}^{-1}(-\mathbf{A} + \boldsymbol{\nu}) \tag{4}$$

(note that, for convenience, we are dropping the references to the independent variables except where clarification is required) where the term $\boldsymbol{\nu} = [\nu_1, \cdots, \nu_p]^\top$ is an input to the linearized plant dynamics. In order for $\mathbf{U}^*$ to be well defined we need the following assumption.

(P1) Plant Assumption: The matrix $\mathbf{B}$ as defined above is nonsingular, i.e., $\mathbf{B}^{-1}$ exists and has bounded norm for all $\mathbf{X} \in S_x$, $t \geq 0$, where $S_x \in \Re^n$ is some compact set of allowable state trajectories. This is equivalent to assuming

$$\sigma_p(\mathbf{B}) \geq \sigma_{\min} > 0 \tag{5}$$

$$\|\mathbf{B}\|_2 = \sigma_1(\mathbf{B}) \leq \sigma_{\max} < \infty \tag{6}$$

where $\sigma_p(\mathbf{B})$ and $\sigma_1(\mathbf{B})$ are, respectively, the smallest and largest singular values of $\mathbf{B}$.

In addition, in order to be able to guarantee state boundedness under state feedback linearization we require P2.

(P2) Plant Assumption: The plant is feedback linearizable by static state feedback; it has a general vector relative degree $\mathbf{r} = [r_1, \cdots, r_p]^\top$ and its zero dynamics are exponentially attractive (please refer to [24] for a review on the concept of zero dynamics and static state feedback of square MIMO systems). We also assume the state vector $\mathbf{X}$ to be available for measurement.

Our goal is to identify the unknown control function (4) using fuzzy systems. Here, we will use generalized T–S fuzzy systems [25] with center average defuzzification. To briefly present the notation, take a fuzzy system denoted by $\tilde{f}(\mathbf{X}, \mathbf{W})$ (in our context, $\mathbf{X}$ could be thought of as the state vector and $\mathbf{W}$ as a vector of possibly exogenous signals). Then, $\tilde{f}(\mathbf{X}, \mathbf{W}) = (\sum_{i=1}^R c_i\mu_i / \sum_{i=1}^R \mu_i)$. Here, singleton fuzzification of the input vectors $\mathbf{X} = [x_1, \cdots, x_n]^\top$, $\mathbf{W} = [w_1, \cdots, w_q]^\top$ is assumed; the fuzzy system has R rules, and μ_i is the value of the membership function for the premise of the ith rule given the inputs $\mathbf{X}$, $\mathbf{W}$. It is assumed that the fuzzy system is constructed in such a way that $0 \leq \mu_i \leq 1$ and $\sum_{i=1}^R \mu_i \neq 0$ for all $\mathbf{X} \in \Re^n$, $\mathbf{W} \in \Re^q$. The parameter c_i is the consequent of the ith rule, which, in this paper, will be taken as a linear combination of Lipschitz continuous functions, $z_k(\mathbf{X}) \in \Re$, $k = 1, \cdots, m-1$, so that $c_i = a_{i,0} + a_{i,1}z_1(\mathbf{X}) + \cdots + a_{i,m-1}z_{m-1}(\mathbf{X})$, $i = 1, \cdots, R$. Define

$$z = \begin{bmatrix} 1\\ z_1(\mathbf{X})\\ \vdots\\ z_{m-1}(\mathbf{X})\end{bmatrix} \in \Re^m$$

$$\zeta^\top = \frac{[\mu_1(\mathbf{X}, \mathbf{W}), \cdots, \mu_R(\mathbf{X}, \mathbf{W})]}{\sum_{i=1}^{R} \mu_i(\mathbf{X}, \mathbf{W})}$$

$$A^\top = \begin{bmatrix} a_{1,0} & a_{1,1} & \cdots & a_{1,m-1}\\ a_{2,0} & a_{2,1} & \cdots & a_{2,m-1}\\ \vdots & \vdots & \ddots & \vdots\\ a_{R,0} & a_{R,1} & \cdots & a_{R,m-1}\end{bmatrix}.$$

Then, the nonlinear equation that describes the fuzzy system can be written as $\tilde{f}(\mathbf{X}, \mathbf{W}) = z^\top(\mathbf{X})A\zeta(\mathbf{X}, \mathbf{W})$ (notice that standard fuzzy systems may be treated as special cases of this more general representation). The T–S model can represent a class of two-layer neural networks and many standard fuzzy systems [8]. Note that while ζ may depend on both $\mathbf{X}$ and $\mathbf{W}$ and is bounded for any value they may take, z depends on $\mathbf{X}$ only. This allows us to impose no restrictions on $\mathbf{W}$ to guarantee boundedness of the fuzzy system.

We will represent the ith component of the ideal control (4), $i = 1, \cdots, p$, as

$$u_i^*(\mathbf{X}, \mathbf{W}, t) = z_i^\top(\mathbf{X})A_i^*\zeta_i(\mathbf{X}, \mathbf{W}) + d_i(\mathbf{X}, \mathbf{W}) + u_{k_i}(t) \tag{7}$$

where $A_i^* \in \Re^{m_i \times R_i}$ is assumed to exist and is defined by

$$A_i^* := \arg\min_{A_i \in \Omega_i}\left[\sup_{\mathbf{X}\in S_x, \mathbf{W}\in S_w, t\geq 0} |z_i^\top A_i\zeta_i + u_{k_i} - u_i^*|\right] \tag{8}$$

and the representation error using optimal parameters (which arises because of the finite number of basis functions used) is bounded, i.e., $d_i(\mathbf{X}, \mathbf{W}) \leq D_i(\mathbf{X}, \mathbf{W})$, where D_i is a known bounding function. We define Ω_i as a compact set within which the matrix of coefficients estimates $A_i(t)$ is allowed to lie, $S_x \subseteq \Re^n$ as the subspace through which the state trajectory may travel under closed-loop control, and $S_w \subseteq \Re^q$ is the subset where the vector $\mathbf{W}$ may lie (notice that we do not restrict the sizes of S_x and S_w; however, the sup in (8) is assumed to exist). As a result of the proof, we will be able to determine that $\mathbf{X}$ actually remains within a compact subset of S_x. Note that the ideal control law (4) is a function not only of the states but also of $\boldsymbol{\nu}$, which may depend on variables other than the states (as will be described below). The vector $\mathbf{W}$ is provided to account for this dependence. The term u_{k_i} represents a *known* part of the ideal control input, which may be available to the designer through knowledge of the plant or expertise. If it is not available, this term may be set equal to zero with all the properties of the adaptive controller still holding. The only restriction on u_{k_i} is that it must be bounded. Note that an appropriate use of u_{k_i} may help to significantly improve the *performance* of the controller, even though in principle it has no effect on stability. Thus, the fuzzy system approximation of u_i^* is given by

$$\hat{u}_i(\mathbf{X}, \mathbf{W}, t) := z_i^\top(\mathbf{X}) A_i(t) \zeta_i(\mathbf{X}, \mathbf{W}) + u_{k_i}(t). \tag{9}$$

The matrix $A_i(t)$ is to be adjusted adaptively on line in order to try to improve the approximation. We define the parameter error matrix, $\Phi_i(t) := A_i(t) - A_i^*$. Let $\hat{\mathbf{U}} := [\hat{u}_1, \cdots, \hat{u}_p]^\top$.

Our objective is to have the plant outputs track a vector of reference trajectories, $\mathbf{Y_m} = [y_{m_1}, \cdots, y_{m_p}]^\top$ on which we make the following assumption.

(R1) Reference Input Assumption: The desired reference trajectories y_{m_i} are r_i times continuously differentiable, with $y_{m_i}, \cdots, y_{m_i}^{(r_i)}$ measurable and bounded for $i = 1, \cdots, p$.

We define the output errors $e_{o_i} := y_{m_i} - y_i$. Define also the error signals

$$e_{s_i} := [k_0^i, \cdots, k_{r_i-2}^i, 1]\,[e_{o_i}, \cdots, e_{o_i}^{(r_i-2)}, e_{o_i}^{(r_i-1)}]^\top$$

and

$$\overline{e}_{s_i} := \dot{e}_{s_i} - e_{o_i}^{(r_i)} = [k_0^i, \cdots, k_{r_i-2}^i]\,[\dot{e}_{o_i}, \cdots, e_{o_i}^{(r_i-1)}]^\top.$$

The coefficients of

$$\frac{1}{\hat{L}_i(s)} := \frac{1}{s^{r_i-1} + k_{r_i-2}^i s^{r_i-2} + \cdots + k_1^i s + k_0^i}$$

$i = 1, \cdots, p$ are picked so that the transfer functions are stable. Let the ith component of the parameter $\boldsymbol{\nu}$ in (4) be given by $\nu_i := y_{m_i}^{(r_i)} + \eta_i e_{s_i} + \overline{e}_{s_i}$, where $\eta_i > 0$ is a constant. Consider the control law

$$\mathbf{U} = \hat{\mathbf{U}} + \mathbf{U_d} \tag{10}$$

where $\mathbf{U_d} := [u_{d_1}, \cdots, u_{d_p}]^\top$ is a control term required to ensure stability that will be defined later.

From (3) and (4), we can derive an expression for the plant output

$$\mathbf{Y^{(r)}} = \mathbf{A} + \mathbf{B}\mathbf{U} = \mathbf{A} + \mathbf{B}(\mathbf{U} - \mathbf{U}^*) + \mathbf{B}\mathbf{U}^* = \boldsymbol{\nu} + \mathbf{B}(\mathbf{U} - \mathbf{U}^*). \tag{11}$$

Then, using the previous definitions, the ith component of the output error dynamics is given by

$$\begin{aligned} e_{o_i}^{(r_i)} &= y_{m_i}^{(r_i)} - y_i^{(r_i)} \\ &= y_{m_i}^{(r_i)} - \nu_i - \sum_{j=1}^{p} \beta_{ij}(u_j - u_j^*) \\ &= -\eta_i e_{s_i} - \overline{e}_{s_i} - \sum_{j=1}^{p} \beta_{ij}(u_j - u_j^*) \end{aligned} \tag{12}$$

so that

$$\dot{e}_{s_i} = -\eta_i e_{s_i} - \sum_{j=1}^{p} \beta_{ij}(u_j - u_j^*). \tag{13}$$

Before proceeding, we need to introduce another set of assumptions on the plant and formalize our assumptions about the control term $\hat{\mathbf{U}}$.

(P3) Plant Assumption: Each entry of $\mathbf{B}$ (besides those on the main diagonal) is bounded by known constants $|\beta_{ij}(\mathbf{X})| \leq \overline{\beta}_{ij}$, $i, j = 1, \cdots, p$, $i \neq j$. We require that the entries in the main diagonal satisfy $0 < \underline{\beta}_{ii} \leq \beta_{ii}(\mathbf{X}) \leq \overline{\beta}_{ii} < \infty$, $i = 1, \cdots, p$ and their derivatives be defined and satisfy $|\dot{\beta}_{ii}(\mathbf{X})| \leq M_{ii}(\mathbf{X})$, $i = 1, \cdots, p$, where $\underline{\beta}_{ii}, \overline{\beta}_{ii}$, and $M_{ii}(\mathbf{X})$ are known bounds. Furthermore, the bounds have to satisfy

$$\frac{1}{\underline{\beta}_{ii}} \sum_{j=1,\, j\neq i}^{p} \overline{\beta}_{ij} < 1, \qquad i = 1, \cdots, p. \tag{14}$$

(C1) Direct Adaptive Control Assumption: Bounding functions $\overline{U}_i(\mathbf{X}, \mathbf{W})$ such that $|z_i^\top(\mathbf{X})\Phi(t)\zeta_i(\mathbf{X}, \mathbf{W})| \leq \overline{U}_i(\mathbf{X}, \mathbf{W})$, $i = 1, \cdots, p$, $\mathbf{X} \in S_x$, $\mathbf{W} \in S_w$ are known and they are continuous functions. Furthermore, the fuzzy systems or neural networks that define the control term $\hat{\mathbf{U}}$ are defined so that the bounding functions of the representation errors $D_i(\mathbf{X}, \mathbf{W})$, $i = 1, \cdots, p$ are continuous.

Note that in P3 the entries of the main diagonal of $\mathbf{B}$ are all assumed positive. This is only to simplify the analysis; the diagonal entries may have any sign as long as they are bounded away from zero and the stability analysis requires only slight modifications to accommodate such a case. In C1, knowledge of the bounding function $\overline{U}_i(\mathbf{X}, \mathbf{W})$ is reasonable since both $z_i(\mathbf{X})$ and $\zeta_i(\mathbf{X}, \mathbf{W})$ are known: a projection algorithm may be employed to guarantee that $A_i(t)$ stays within the compact set Ω_i of allowable parameters. Then, an upper estimate of $||\Phi_i(t)||$ can be computed, and $\overline{U}_i$ can be defined.

Consider the function

$$V_i = \frac{1}{2\beta_{ii}} e_{s_i}^2 + \frac{1}{2}\,\mathbf{tr}\big(\Phi_i^\top Q_{u_i} \Phi_i\big) \tag{15}$$

with $Q_{u_i} \in \Re^{m_i \times m_i}$ positive definite and diagonal. This function quantifies both the tracking error for the ith plant output and the approximation error for the parameter estimates of the ith term of (4). Taking the derivative of (15) yields

$$\dot{V}_i = \frac{1}{\beta_{ii}} e_{s_i}\dot{e}_{s_i} + \mathbf{tr}\Big(\Phi_i^\top Q_{u_i} \dot{\Phi}_i\Big) - \frac{\dot{\beta}_{ii}}{2\beta_{ii}^2} e_{s_i}^2 \tag{16}$$

$$= \frac{1}{\beta_{ii}} e_{s_i} \left(-\eta_i e_{s_i} - \sum_{j=1}^{p} \beta_{ij}(u_j - u_j^*) \right) + \mathbf{tr}\left(\Phi_i^\top Q_{u_i} \dot{\Phi}_i\right) - \frac{\dot{\beta}_{ii}}{2\beta_{ii}^2} e_{s_i}^2. \tag{17}$$

Define the adaptation law for the T–S fuzzy system or neural network as

$$\dot{A}_i := Q_{u_i}^{-1} z_i \zeta_i^\top e_{s_i} \tag{18}$$

so that applying the properties of the trace operator and the fact that $\dot{\Phi}_i = \dot{A}_i$ we obtain $\mathbf{tr}(\Phi_i^\top Q_{u_i} \dot{\Phi}_i) = z_i^\top \Phi_i \zeta_i \overline{e}_{s_i}$. Noting that $u_i - u_i^* = u_{d_i} + z_i^\top \Phi_i \zeta_i - d_i$, we get

$$\dot{V}_i = -\frac{\eta_i}{\beta_{ii}} e_{s_i}^2 + \frac{e_{s_i}}{\beta_{ii}} \cdot \left[-\sum_{j=1}^{p} \beta_{ij} u_{d_j} + \sum_{j=1}^{p} \beta_{ij} d_j - \sum_{j=1, j\neq i}^{p} \beta_{ij} z_j^\top \Phi_j \zeta_j \right] - \frac{\dot{\beta}_{ii}}{2\beta_{ii}^2} e_{s_i}^2 \tag{19}$$

$$\dot{V}_i \le -\frac{\eta_i}{\beta_{ii}} e_{s_i}^2 - e_{s_i} u_{d_i} + |e_{s_i}| \cdot \left[\sum_{j=1}^{p} \frac{\overline{\beta}_{ij}}{\underline{\beta}_{ii}} D_j + \sum_{j=1, j\neq i}^{p} \frac{\overline{\beta}_{ij}}{\underline{\beta}_{ii}} \left(|u_{d_j}| + |z_j^\top \Phi_j \zeta_j| \right) \right] + \frac{|\dot{\beta}_{ii}|}{2\beta_{ii}^2} e_{s_i}^2. \tag{20}$$

Define

$$\sigma_i := \sum_{j=1}^{p} \frac{\overline{\beta}_{ij}}{\underline{\beta}_{ii}} D_j + \sum_{j=1, j\neq i}^{p} \frac{\overline{\beta}_{ij}}{\underline{\beta}_{ii}} \overline{U}_j$$

$$\rho_i := e_{s_i} \left(\frac{M_{ii}(\mathbf{X})}{2\underline{\beta}_{ii}^2} \right).$$

Given these definitions, let

$$u_{d_i} := \mathrm{sgn}(e_{s_i}) \left(\sigma_i + \sum_{j=1, j\neq i}^{p} \frac{\overline{\beta}_{ij}}{\underline{\beta}_{ii}} U_{\max} \right) + \rho_i \tag{21}$$

where we need to derive an expression for $U_{\max}$ such that $|u_{d_i}| \le U_{\max}$, $i = 1, \cdots, p$. From (21) we have

$$|u_{d_i}| \le |\sigma_i| + |\rho_i| + U_{\max} \sum_{j=1, j\neq i}^{p} \frac{\overline{\beta}_{ij}}{\underline{\beta}_{ii}} \le U_{\max}. \tag{22}$$

It follows that if we choose

$$U_{\max}(t) \ge \max_{i=1,\cdots,p} \left[\frac{|\sigma_i| + |\rho_i|}{1 - \sum_{j=1, j\neq i}^{p} \frac{\overline{\beta}_{ij}}{\underline{\beta}_{ii}}} \right] \tag{23}$$

and if P3 holds, then in fact $|u_{d_i}| \le U_{\max}$, $i = 1, \cdots, p$. Using (21) we can now establish

$$\dot{V}_i \le -\frac{\eta_i}{\beta_{ii}} e_{s_i}^2. \tag{24}$$

We are now ready to present our main result and give its proof.

Theorem 1—Stability and Tracking Results Using MIMO Direct Adaptive Control: If the reference input assumption R1 holds, the plant assumptions P1, P2, and P3 hold and the control law is defined by (10) with the control assumption C1 and the adaptive laws (18) are used, *then* the following holds:

1) The plant states as well as its outputs and their derivatives $y_i, \cdots, y_i^{(r_i-1)}$, $i = 1, \cdots, p$ are bounded.
2) The control signals are bounded, i.e., $\|\mathbf{U}\| \in \mathcal{L}_\infty$ ($\mathcal{L}_\infty = \{\phi(t)\colon \sup_t |\phi(t)| < \infty\}$).
3) The magnitudes of the output errors e_{o_i} decrease at least asymptotically to zero, i.e., $\lim_{t\to\infty} e_{o_i} = 0$, $i = 1, \cdots, p$.

Proof of Theorem 1: To show part 1, consider the Lyapunov candidate

$$V := \sum_{i=1}^{p} V_i. \tag{25}$$

The above analysis guarantees

$$\dot{V} \le -\sum_{i=1}^{p} \frac{\eta_i}{\beta_{ii}} e_{s_i}^2 \tag{26}$$

so V is a positive definite function with negative semidefinite derivative. This implies that $V_i \in \mathcal{L}_\infty$; therefore, $e_{s_i}, \overline{e}_{s_i} \in \mathcal{L}_\infty$ and $\|\Phi_i\| \in \mathcal{L}_\infty$ for $i = 1, \cdots, p$ (notice that this analysis alone does not guarantee $A_i \in \Omega_i$ for all time; rather, a projection algorithm should be used to achieve this). From the definition of e_{s_i} we have $e_{o_i}^{(j)} = \hat{G}_i^j(s) e_{s_i}$, where $\hat{G}_i^j(s) := s^j/\hat{L}_i(s)$, $j = 0, \cdots, r_i - 1$. Since by definition $\hat{G}_i^j$ is stable, $e_{o_i}^{(j)} \in \mathcal{L}_\infty$, $j = 0, \cdots, r_i - 1$ and since by assumption (R1) the signals $y_{m_i}^{(r_i)}$ are bounded, we conclude that $y_i, \cdots, y_i^{(r_i-1)} \in \mathcal{L}_\infty$, $i = 1, \cdots, p$.

With the outputs bounded and together with assumption P2 we have that the states $x_1, \cdots, x_n$ are bounded [23], which implies that the state trajectories are limited to a bounded subset of S_x. Let $\overline{S}_x$ be the compact ball of minimum radius that contains the bounded subset of state trajectories. Since ζ_i is continuous and z_i is Lipschitz continuous by definition in S_x, then they are uniformly continuous and, therefore, bounded on $\overline{S}_x$; and given that u_{k_i} is bounded, we have $\hat{u}_i \in \mathcal{L}_\infty$, $i = 1, \cdots, p$. Since $\overline{U}_i$ is defined as a continuous function, and D_i is assumed continuous for all $\mathbf{X} \in S_x$, $\mathbf{W} \in S_w$, both are bounded on $\overline{S}_x$, so $\sigma_i \in \mathcal{L}_\infty$, and $\rho_i \in \mathcal{L}_\infty$ because $e_{s_i} \in \mathcal{L}_\infty$ from part 1. This implies $U_{\max} \in \mathcal{L}_\infty$, so $u_{d_i} \in \mathcal{L}_\infty$, $i = 1, \cdots, p$ by construction. Hence, $\|\mathbf{U}\| \in \mathcal{L}_\infty$.

To prove part 3 we notice that from (26)

$$\int_0^\infty \sum_{i=1}^{p} \frac{\eta_i}{\beta_{ii}} e_{s_i}^2 \, dt \le -\int_0^\infty \dot{V}\, dt \tag{27}$$

$$= V(0) - V(\infty) < \infty. \tag{28}$$

This establishes that $e_{s_i} \in \mathcal{L}_2, i = 1, \cdots, p$ ($\mathcal{L}_2 = \{\phi(t)\colon \int_0^\infty \phi^2(t)\, dt < \infty\}$). Having determined that $e_{o_i}^j \in \mathcal{L}_\infty$, $j = 1, \cdots, r_i - 1$, it follows that $\dot{e}_{s_i} \in \mathcal{L}_\infty$, so e_{s_i} is uniformly continuous. Since $e_{s_i} \in \mathcal{L}_2 \bigcap \mathcal{L}_\infty$ and $\dot{e}_{s_i} \in \mathcal{L}_\infty$, by Barbalat's Lemma we have asymptotic stability of e_{s_i} (i.e., $\lim_{t\to\infty} e_{s_i} = 0$), which implies asymptotic stability of e_{o_i}

(i.e., $\lim_{t\to\infty} e_{o_i} = 0$), for $i = 1, \cdots, p$. Notice that although assumption P1 is not used explicitly in the proof, it is still necessary in order to guarantee the *existence* of $\mathbf{U}^*$ without which the argument is not sound. □

Remark: Note that although in principle the choice of the vector $\mathbf{W}$ is arbitrary, a typical choice may be an error vector, i.e., $\mathbf{W} = \mathbf{Y} - \mathbf{Y_m}$, or some other function of the plant outputs and the reference model outputs. In this way, as a result of the proof, we also obtain that $\mathbf{W}$ remains within a bounded subset of S_w.

III. Indirect Adaptive Control

Here we consider, again, the class of plants defined in (1). If assumptions P1 and P2 of Section II are satisfied, then we may rewrite the IO form of the plant as

$$\begin{bmatrix} y_1^{(r_1)} \\ \vdots \\ y_p^{(r_p)} \end{bmatrix} = \underbrace{\begin{bmatrix} \alpha_1 + \alpha_{1_k} \\ \vdots \\ \alpha_p + \alpha_{p_k} \end{bmatrix}}_{\mathbf{A}(\mathbf{X},t)} + \underbrace{\begin{bmatrix} \beta_{11} + \beta_{11_k} & \cdots & \beta_{1p} + \beta_{1p_k} \\ \vdots & \ddots & \vdots \\ \beta_{p1} + \beta_{p1_k} & \cdots & \beta_{pp} + \beta_{pp_k} \end{bmatrix}}_{\mathbf{B}(\mathbf{X},t)} \underbrace{\begin{bmatrix} u_1 \\ \vdots \\ u_p \end{bmatrix}}_{\mathbf{U}(t)} \quad (29)$$

where $\alpha_{i_k}(t)$ and $\beta_{ij_k}(t)$ are known components of the plant's dynamics (that may depend on the state) or exogenous time dependent signals with the only constraint that they have to be bounded for all $t \geq 0$. Throughout the following analysis they may be set equal to zero for all t; however, as in the direct adaptive case a good choice of these known functions may help improve the *performance* of the controller. The functions $\alpha_i(\mathbf{X})$ and $\beta_{ij}(\mathbf{X})$ represent unknown nonlinear dynamics of the plant.

Again consider the ideal feedback linearizing control law (4), where the term $\boldsymbol{\nu}$ will be redefined below. Our goal is to identify the unknown functions α_i and β_{ij} using fuzzy systems (or neural networks) in order to indirectly approximate the ideal control law $\mathbf{U}^*$. Let the fuzzy system be a T–S form with center average defuzzification as in Section II. We rewrite

$$\alpha_i(\mathbf{X}) = z_{\alpha_i}^\top A_{\alpha_i}^* \zeta_{\alpha_i} + d_{\alpha_i}(\mathbf{X}) \quad (30)$$

$$\beta_{ij}(\mathbf{X}) = z_{\beta_{ij}}^\top A_{\beta_{ij}}^* \zeta_{\beta_{ij}} + d_{\beta_{ij}}(\mathbf{X}) \quad (31)$$

where $A_{\alpha_i}^* \in \Re^{m_{\alpha_i} \times p_{\alpha_i}}$ and $A_{\beta_{ij}}^* \in \Re^{m_{\beta_{ij}} \times p_{\beta_{ij}}}$ are defined by

$$A_{\alpha_i}^* := \arg \min_{A_{\alpha_i} \in \Omega_{\alpha_i}} \left[\sup_{\mathbf{X} \in S_x} \left| z_{\alpha_i}^\top A_{\alpha_i} \zeta_{\alpha_i} - \alpha_i \right| \right] \quad (32)$$

$$A_{\beta_{ij}}^* := \arg \min_{A_{\beta_{ij}} \in \Omega_{\beta_{ij}}} \left[\sup_{\mathbf{X} \in S_x} \left| z_{\beta_{ij}}^\top A_{\beta_{ij}} \zeta_{\beta_{ij}} - \beta_{ij} \right| \right]. \quad (33)$$

Note that we are assuming the ability to specify fuzzy systems in such a way that the representation errors using optimal parameters (which arise because of the finite number of basis functions used) are bounded, i.e., $d_{\alpha_i}(\mathbf{X}) \leq D_{\alpha_i}(\mathbf{X})$, $d_{\beta_{ij}}(\mathbf{X}) \leq D_{\beta_{ij}}(\mathbf{X})$, where $D_{\alpha_i}(\mathbf{X})$ and $D_{\beta_{ij}}(\mathbf{X})$ are known bounding functions. We require the representation errors $D_{\beta_{ij}}(\mathbf{X})$ to be small (later we will provide an explicit condition as to how small they have to be), which means that the matrix $\mathbf{B}$ can, ideally, be well approximated by our fuzzy systems (or neural networks) using optimal parameters. Our adaptive controller's stability will not, however, depend on its ability to identify these optimal parameters.

The compact set $S_x \subseteq \Re^n$ is defined as before and Ω_{α_i}, $\Omega_{\beta_{ij}}$ are compact sets within which the parameter matrices estimates $A_{\alpha_i}(t)$ and $A_{\beta_{ij}}(t)$ are allowed to lie. Thus, the fuzzy system approximations of $\alpha_i(\mathbf{X})$ and $\beta_{ij}(\mathbf{X})$ are given by

$$\hat{\alpha}_i(\mathbf{X}) := z_{\alpha_i}^\top A_{\alpha_i} \zeta_{\alpha_i} \quad (34)$$

$$\hat{\beta}_{ij}(\mathbf{X}) := z_{\beta_{ij}}^\top A_{\beta_{ij}} \zeta_{\beta_{ij}}. \quad (35)$$

The matrices $A_{\alpha_i}(t)$ and $A_{\beta_{ij}}(t)$ are to be adjusted adaptively on line in order to try to improve the approximation.

We define the output errors e_{o_i} and error signals e_{s_i} and $\overline{e}_{s_i}$ as in Section II. Consider the control law $\mathbf{U} := \mathbf{U_{ce}}$, where $\mathbf{U_{ce}} = [u_{ce_1}, \cdots, u_{ce_p}]^\top$ is a *certainty equivalence* control term. Define the matrix $\hat{\mathbf{B}} := [\hat{\beta}_{ij}(\mathbf{X}) + \beta_{ij_k}(t)]$, $i, j = 1, \cdots, p$. $\hat{\mathbf{B}}$ is an approximation of the ideal and unknown matrix $\mathbf{B}$. Furthermore, let $[b_{ij}] := \hat{\mathbf{B}}^{-1}$, $i, j = 1, \cdots, p$ be a matrix of the elements of the inverse. We need to ensure that $\hat{\mathbf{B}}^{-1}$ exists for all $\mathbf{X} \in S_x$ and $t \geq 0$. If the sets $\Omega_{\beta_{ij}}$ are constructed such that

$$\sigma_p(\hat{\mathbf{B}}) \geq \sigma_{\min} \quad (36)$$

$$\sigma_1(\hat{\mathbf{B}}) \leq \sigma_{\max} \quad (37)$$

for all $\mathbf{X} \in S_x$, then as long as the matrices $A_{\beta_{ij}}$ remain within $\Omega_{\beta_{ij}}$, respectively, we can guarantee that $\hat{\mathbf{B}}^{-1}$ exists (this can be achieved using a projection algorithm). Note that if we knew the matrix $\mathbf{B}$ to be, for instance, strictly diagonally dominant [as required by the Levy–Desplanques theorem (see [26] for an explanation of this and other invertibility results)] with known lower bounds for the main diagonal entries, we could relax the conditions on the sets $\Omega_{\beta_{ij}}$ by applying instead a projection algorithm that kept $\hat{\mathbf{B}}$ in a strictly diagonally dominant form similar to $\mathbf{B}$ to ensure it is invertible.

In order to cancel the unknown parameter errors we use the following adaptive laws:

$$\begin{aligned} \dot{A}_{\alpha_i} &= -Q_{\alpha_i}^{-1} z_{\alpha_i} \zeta_{\alpha_i} e_{s_i} \\ \dot{A}_{\beta_{ij}} &= -Q_{\beta_{ij}}^{-1} z_{\beta_{ij}} \zeta_{\beta_{ij}} e_{s_i} u_{ce_j} \end{aligned} \quad (38)$$

with $Q_{\alpha_i} \in \Re^{m_{\alpha_i} \times m_{\alpha_i}}$, $Q_{\beta_{ij}} \in \Re^{m_{\beta_{ij}} \times m_{\beta_{ij}}}$ positive definite and diagonal.

Now we can write an expression for $\mathbf{U_{ce}}$

$$\mathbf{U_{ce}} := \hat{\mathbf{B}}^{-1} \left(-\hat{\mathbf{A}} + \boldsymbol{\nu} \right) \quad (39)$$

where $\hat{\mathbf{A}} := [\hat{\alpha}_1 + \alpha_{1_k}, \cdots, \hat{\alpha}_p + \alpha_{p_k}]^\top$. Here $\boldsymbol{\nu}(t) = [\nu_1(t), \cdots, \nu_p(t)]^\top$ is chosen to provide stable tracking and to allow for robustness to parameter uncertainty. Namely, let

$$\nu_i(t) := y_{m_i}^{(r_i)} + \eta_i e_{s_i} + \overline{e}_{s_i} + D_{\alpha_i}(\mathbf{X}) \mathrm{sgn}(e_{s_i}) + U_{\max}(\mathbf{X}) \mathrm{sgn}(e_{s_i}) \sum_{j=1}^{p} D_{\beta_{ij}}(\mathbf{X}) \quad (40)$$

where $\eta_i > 0$ is a design parameter and $U_{\max}(\mathbf{X})$ is a function chosen so that $|u_{ce_i}| \leq U_{\max}(\mathbf{X})$, $i = 1, \cdots, p$. It can be shown that the choice

$$U_{\max}(\mathbf{X}) \geq \max_{i=1,\cdots,p} \left[\frac{a_i(\mathbf{X})}{1 - c_i(\mathbf{X})} \right] \tag{41}$$

satisfies the requirement, where $a_i(\mathbf{X}) := \sum_{j=1}^{p} |b_{ij}|[|\hat{\alpha}_j| + |\alpha_{j_k}| + |y_{m_j}^{(r_j)}| + \eta_j |e_{s_j}| + |\bar{e}_{s_j}| + D_{\alpha_j}]$ and $c_i(\mathbf{X}) := \sum_{j=1}^{p} |b_{ij}| \sum_{l=1}^{p} D_{\beta_{jl}}$. It should be noted that for many classes of plants, each β_{ij} is a smooth function easily represented by a fuzzy system. For example, if each β_{ij} may be expressed as a constant, then $D_{\beta_{ij}} = 0$ for all i, j since a fuzzy system may exactly represent a constant on a compact set. This would remove the need for the $U_{\max}(\mathbf{X})$ term to be included in (40).

At this point we need to formalize our general assumption about the controller.

(C2) Indirect Adaptive Control Assumption: The fuzzy systems (or neural networks) that define the approximations (34) and (35) are defined so that $D_{\alpha_i}(\mathbf{X}) \in \mathcal{L}_\infty$, $D_{\beta_{ij}}(\mathbf{X}) \in \mathcal{L}_\infty$, for all $\mathbf{X} \in S_x \subseteq \Re^n$, $i, j = 1, \cdots, p$. Furthermore, the ideal representation errors $D_{\beta_{ij}}(\mathbf{X})$ are small enough so that we have $0 \leq c_i(\mathbf{X}) < 1$ for all $\mathbf{X} \in S_x$, $i = 1, \cdots, p$.

Notice that the maximum sizes of $D_{\beta_{ij}}$ that satisfy C2 can be found since from (36) we have $|b_{ij}(\mathbf{X})| \leq \|\hat{\mathbf{B}}^{-1}\|_2 \leq (1/\sigma_{\min})$ and as long as assumption C2 is satisfied, we ensure that $|u_{ce_i}| \leq U_{\max}(\mathbf{X})$, $i = 1, \cdots, p$ as desired.

We have completely specified the signals that compose the control vector $\mathbf{U}$ and now we state our main result; its proof is omitted but follows ideas used in Section II (please refer to [22] for the details of the proof).

Theorem 2—Stability and Tracking Results Using MIMO Indirect Adaptive Control: If the reference input assumption R1 holds, the plant assumptions P1 and P2 hold and the control law is defined by (39) with the control assumption C2, *then* the following holds:

1) The plant states as well as its outputs and their derivatives $y_i, \cdots, y_i^{(r_i-1)}$, $i = 1, \cdots, p$ are bounded.
2) The control signals are bounded, i.e., $u_{ce_i} \in \mathcal{L}_\infty$, $i = 1, \cdots, p$.
3) The magnitudes of the output errors e_{o_i} decrease at least asymptotically to zero, i.e., $\lim_{t\to\infty} e_{o_i} = 0$, $i = 1, \cdots, p$.

IV. Applications

A. Illustrative Example

Consider the nonlinear differential equation given by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_1 + x_2^2 + x_3 \\ x_1 + 2x_2 + 3x_3x_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3u_1 + u_2 \\ u_1 + 2(2 + 0.5\sin(x_1))u_2 \end{bmatrix}. \tag{42}$$

TABLE I
RULE BASE

	$\mathbf{F_1^1}$	$\mathbf{F_2^1}$	$\mathbf{F_3^1}$	$\mathbf{F_4^1}$	$\mathbf{F_5^1}$	$\mathbf{F_6^1}$	$\mathbf{F_7^1}$	$\mathbf{F_8^1}$	$\mathbf{F_9^1}$
$\mathbf{F_1^2}$	c_1^k	c_1^k	c_1^k	c_1^k	c_1^k	c_2^k	c_3^k	c_4^k	c_5^k
$\mathbf{F_2^2}$	c_1^k	c_1^k	c_1^k	c_1^k	c_2^k	c_3^k	c_4^k	c_5^k	c_6^k
$\mathbf{F_3^2}$	c_1^k	c_1^k	c_1^k	c_2^k	c_3^k	c_4^k	c_5^k	c_6^k	c_7^k
$\mathbf{F_4^2}$	c_1^k	c_1^k	c_2^k	c_3^k	c_4^k	c_5^k	c_6^k	c_7^k	c_8^k
$\mathbf{F_5^2}$	c_1^k	c_2^k	c_3^k	c_4^k	c_5^k	c_6^k	c_7^k	c_8^k	c_9^k
$\mathbf{F_6^2}$	c_2^k	c_3^k	c_4^k	c_5^k	c_6^k	c_7^k	c_8^k	c_9^k	c_9^k
$\mathbf{F_7^2}$	c_3^k	c_4^k	c_5^k	c_6^k	c_7^k	c_8^k	c_9^k	c_9^k	c_9^k
$\mathbf{F_8^2}$	c_4^k	c_5^k	c_6^k	c_7^k	c_8^k	c_9^k	c_9^k	c_9^k	c_9^k
$\mathbf{F_9^2}$	c_5^k	c_6^k	c_7^k	c_8^k	c_9^k	c_9^k	c_9^k	c_9^k	c_9^k

Notice that these are coupled nonlinear dynamics. The $\mathbf{B}$ matrix is not constant but contains a bounded function of the states. We are interested in the outputs $y_1 = x_1$ and $y_2 = x_3$. It is easily verified that this system has a vector relative degree of $[2, 1]^\top$. We define the error equations as $e_{s_1} = e_{o_1} + \dot{e}_{o_1}$ and $e_{s_2} = e_{o_2}$ with the output errors defined appropriately. We want the outputs of the system to track the reference vector

$$[Y_{m_1}(s), Y_{m_2}(s)]^\top = \left[\frac{R_1(s)}{(s+1)^2}, \frac{R_2(s)}{s+1} \right]^\top$$

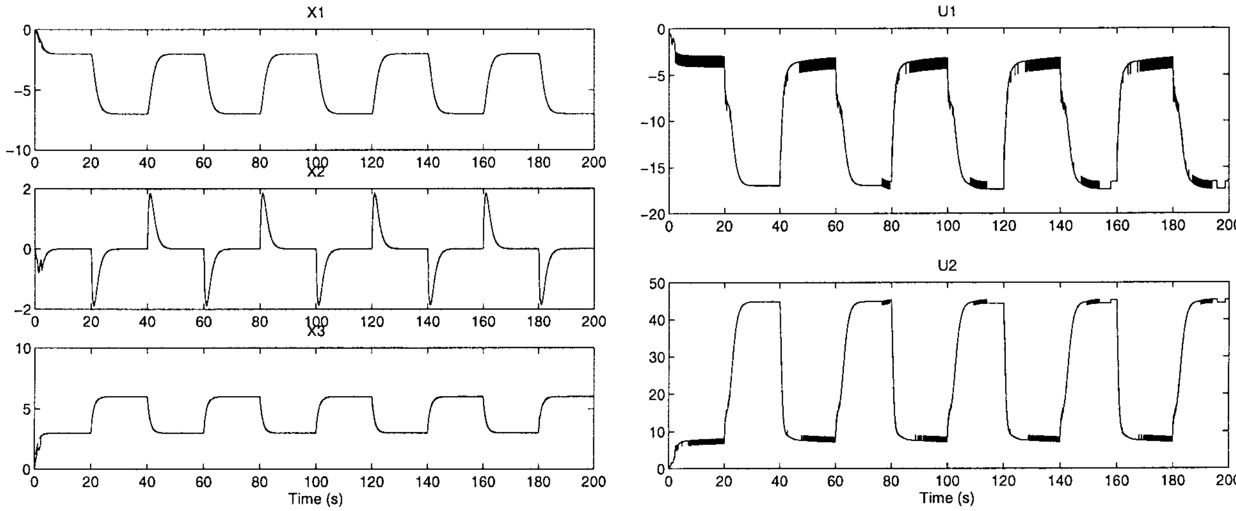
where $R_1(s) = \mathcal{L}\{r_1(t)\}$ and $R_2(s) = \mathcal{L}\{r_2(t)\}$ ($\mathcal{L}\{\cdot\}$ is the Laplace transform operator). Thus, $\dot{e}_{o_1}$ is computed from $\dot{y}_{m_1}$ and x_2.

We use two T–S fuzzy systems to produce $\hat{u}_1$ and $\hat{u}_2$ and we set the "known" controller terms $u_{k_i} = 0, i = 1, 2$. Both fuzzy systems have e_{o_1} and e_{o_2} as their inputs (so here $\mathbf{W} = [e_{o_1}, e_{o_2}]^\top$) and we let $z_k^\top = [1, x_1, x_2, x_3]$, $k = 1, 2$. Both fuzzy systems have nine triangular membership functions for each of the two input universes of discourse, uniformly distributed over the interval $[-1, 1]$ with 50% overlap (we use scaling gains to normalize the inputs to this interval). We saturate the outermost membership functions and the output is computed using center average defuzzification. Both systems' coefficient matrices A_1 and A_2 are initialized with zeroes and they utilize the rule base shown in Table I. The labels F_i^j denote the ith fuzzy set for the jth input, where $i = 1$ corresponds to the leftmost and $i = 9$ to the rightmost fuzzy set. Each entry of the table corresponds to one output function c_i^k, $i = 1, \cdots, 9$, where $c_i^k = z_k^\top \underline{a}_i^k$, and $\underline{a}_i^k$ is the ith column of A_k, $k = 1, 2$. As an example, consider the rule for c_3^k that is inside of a box in the table

$$\textbf{If } e_{o_1} \text{ is } F_3^1 \textbf{ and } e_{o_2} \text{ is } F_5^2 \textbf{ then } c_3^k = z_k^\top \underline{a}_3^k$$

where we evaluate the **and** operation using minimum. Note that μ_i, $i = 1, \cdots, 9$, is the result of evaluating the *premise* of the ith rule.

From the plant's equation we choose the bounds $\overline{\beta}_{11} = 3.3$, $\underline{\beta}_{11} = 2.7$, $\overline{\beta}_{22} = 5.3$, $\underline{\beta}_{22} = 2.7$, $\underline{\beta}_{21} = 1.3$, $\underline{\beta}_{12} = 1.3$, $M_{11} = 0.0$, and $M_{22} = x_2$ (these two bounds are obtained by differentiating the diagonal entries of the matrix $\mathbf{B}$). Also, the fuzzy system approximation error bounds are chosen as $D_1 = 0.1$, $D_2 = 0.1$ (note that this choice is not readily apparent from the definitions of the fuzzy systems; rather, the



(a) (b)

Fig. 1. (a) System states (solid lines) and reference model outputs (dashed lines). (b) Control inputs.

bounds are found through a trial and error procedure). Since we know the vectors $z_k^\top$ and $\zeta_k^\top$, $k = 1, 2$, an easy way to compute the bounding functions $\overline{U}_1$ and $\overline{U}_2$ is to use fuzzy systems that have the same structure as the ones used for $\hat{u}_1$ and $\hat{u}_2$. Initially, we chose the entries in their coefficient matrices to be large, so that they would bound the values taken by the $\hat{u}_1$ and $\hat{u}_2$ fuzzy systems. We found, however, that this created high-amplitude and high-frequency oscillations of the control signals (due to the term $\mathbf{U_d}$), which are undesirable. After some tuning we determined that setting the $\overline{U}_1$ and $\overline{U}_2$ fuzzy system's coefficients to 0.1 gave us stable behavior, good tracking, and much smoother control signals. Finally, the adaptation gains were chosen as $Q_{u_1} = Q_{u_2} = 2.4I$, where I is a 2×2 identity matrix.

In Fig. 1, we observe the results for direct adaptive control on this system. We used a fourth-order Runge–Kutta numerical approximation to the differential equation solution with a step size of 0.001. The reference inputs $r_1(t)$ and $r_2(t)$ are chosen as square waves, and the corresponding reference model outputs are plotted in Fig. 1(a) with dashed lines but are hard to see since x_1 and x_3 track them closely. In Fig. 1(b), we observe the applied control inputs.

B. Application to a Two-Link Robot Arm

Next, we consider direct adaptive control of a two-degree of freedom robot arm. This system does not satisfy assumption P3 because, as we will see, the matrix multiplying the input vector $\mathbf{U}$ contains functions of the states (similar to the example in the previous section). However, in some regions the bounds for the entries do not satisfy the diagonal dominance condition. Nevertheless, our simulation results show that the method seems to be able to provide stable tracking with adequate performance; furthermore, the controller is able to compensate for an "unknown" change in system parameters, which represents the situation where the robot picks up an object after some time of nominal operation (i.e., when the robot is not holding any object).

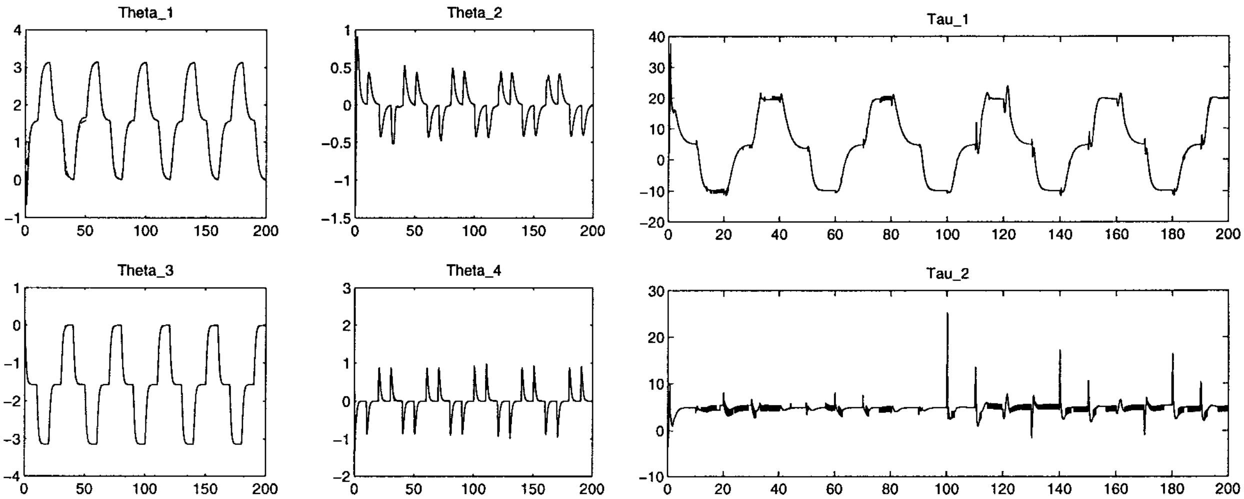
The robot arm consists of two links—the first one mounted on a rigid base by means of a frictionless hinge and the second mounted at the end of link one by means of a frictionless ball bearing. The inputs to the system are the torques τ_1 and τ_2 applied at the joints. The outputs are the joint angles θ_1 and θ_2. A mathematical model of this system can be derived using Lagrangian equations and is given by

$$\underbrace{\begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} -h\dot{\theta}_2 & -h\dot{\theta}_1 - h\dot{\theta}_2 \\ h\dot{\theta}_1 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} \quad (43)$$

where

$$\begin{aligned} H_{11} &= I_1 + I_2 + m_1 l_{c_1}^2 m_2 \left[l_1^2 + l_{c_2}^2 + 2 l_1 l_{c_2} \cos(\theta_2)\right] \\ &\quad + m_3 \left[l_1^2 + l_2^2 + 2 l_1 l_2 \cos(\theta_2)\right] \\ H_{22} &= I_2 + m_2 l_{c_2}^2 + m_3 l_2^2 \\ H_{12} &= H_{21} = I_2 + m_2 \left[l_{c_2}^2 + l_1 l_{c_2} \cos(\theta_2)\right] \\ &\quad + m_3 \left[l_2^2 + l_1 l_2 \cos(\theta_2)\right] \\ h &= m_2 l_1 l_{c_2} \sin(\theta_2) \\ g_1 &= m_1 l_{c_1} g \cos(\theta_1) + m_2 g [l_{c_2} \cos(\theta_1 + \theta_2) + l_1 \cos(\theta_1)] \\ g_2 &= m_2 l_{c_2} g \cos(\theta_1 + \theta_2). \end{aligned}$$

The matrix $\mathbf{H}$ can be shown to be positive definite and, therefore, always invertible. In our simulation, we use the following parameter values: $m_1 = 1.0$ kg, mass of link one; $m_2 = 1.0$ kg, mass of link two; $l_1 = 1.0$ m, length of link one; $l_2 = 1.0$ m, length of link two; $l_{c_1} = 0.5$ m, distance from the joint of link one to its center of gravity; $l_{c_2} = 0.5$ m, distance from the joint of link two to its center of gravity; $I_1 = 0.2$ kg m^2, lengthwise centroidal inertia of link one; and $I_2 = 0.2$ kg m^2, lengthwise centroidal inertia of link two. The mass m_3, initially set equal to zero, represents the mass of an object at the end of the link. After 100 s of operation, the robot "picks up" an object of mass $m_3 = 3.0$ kg.



(a) (b)

Fig. 2. (a) System states (solid lines) and reference model outputs (dashed lines). (b) Control inputs.

We can rewrite the system dynamics in IO form

$$\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} = \frac{1}{H_{11}H_{22}-H_{12}H_{21}} \cdot \left[\begin{bmatrix} H_{22}h\dot{\theta}_2(2\dot{\theta}_1+\dot{\theta}_2)+H_{12}h\dot{\theta}_1^2 - H_{22}g_1+H_{12}g_2 \\ -H_{21}h\dot{\theta}_2(2\dot{\theta}_1+\dot{\theta}_2) - H_{11}h\dot{\theta}_1^2 + H_{21}g_1 - H_{11}g_2\end{bmatrix} + \begin{bmatrix}H_{22} & -H_{12}\\ -H_{21} & H_{11}\end{bmatrix}\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}\right].$$

We observe that the input vector is multiplied by $\mathbf{H}^{-1}$, which contains the function $\cos(\theta_2)$. For some values of θ_2, $\mathbf{H}^{-1}$ is not diagonally dominant and, thus, does not satisfy assumption P3 (note, e.g., that for some θ_2 values, $H_{22} < |H_{12}|$). However, we found that not only was it possible to make the direct adaptive method work, but also that it offered relatively good performance and the ability to handle system parameter changes.

We would like the outputs θ_1 and θ_2 to track desired reference angles, which are obtained from the reference model vector

$$[Y_{m_1}(s), Y_{m_2}(s)]^\top = \left[\frac{0.75^2 R_1(s)}{(s+0.75)^2}, \frac{1.5^2 R_2(s)}{(s+1.5)^2}\right]^\top$$

where $R_1(s) = \mathcal{L}\{r_1(t)\}$ and $R_2(s) = \mathcal{L}\{r_2(t)\}$. Clearly, the system has a vector relative degree $[2, 2]^\top$, so letting $e_{o_1} = y_{m_1} - \theta_1$ and $e_{o_2} = y_{m_2} - \theta_2$, we define the error equations $e_{s_1} = e_{o_1} + \dot{e}_{o_1}$ and $e_{s_2} = e_{o_2} + \dot{e}_{o_2}$. The error derivatives are available since the reference inputs are twice differentiable and the angle derivatives $\dot{\theta}_1$ and $\dot{\theta}_2$ are plant states.

When designing our fuzzy systems we assumed that there is no strong cross-coupling between the inputs and outputs, which greatly simplifies the design: we chose e_{o_1} and $\dot{e}_{o_1}$ as inputs to the fuzzy system for τ_1 (so for this fuzzy system $\mathbf{W} = [e_{o_1}, \dot{e}_{o_1}]^\top$), and e_{o_2} and $\dot{e}_{o_2}$ as inputs for τ_2. Both fuzzy systems have $z_i^\top = [1, \dot{\theta}_1, \ddot{\theta}_1, \dot{\theta}_2, \ddot{\theta}_2]$ and are otherwise structurally identical to the systems defined in Section IV-A. Here also, we initialized the coefficient matrices of both systems with zeros. Since, as mentioned before, the system does not satisfy assumption P3 for all θ_2, the way to choose the bounds for the $\mathbf{H}^{-1}$ matrix entries is not clear. In view of this, we took a pragmatic approach where we first picked bounds that were as close as possible to the real bounds and yet satisfied assumption P3: letting $\mathbf{B} = \mathbf{H}^{-1}$, substitution of the numerical values of the parameters shows that taking into account both values of m_3, $1.1 < \beta_{11} < 1.2$, $2.3 < \beta_{12}, \beta_{21} < 2.5$, and $0.7 < \beta_{22} < 7.3$. Thus, we picked $\overline{\beta}_{11} = 1.3$, $\underline{\beta}_{11} = 1.1$, $\overline{\beta}_{22} = 2.3$, $\underline{\beta}_{22} = 2.5$, $\underline{\beta}_{21} = 2.3$, and $\underline{\beta}_{12} = 2.3$. This choice resulted in somewhat acceptable behavior, but with highly oscillatory control signals. Therefore, we decided to tune these bounds even though the theoretical assumptions were violated. We found that reducing the size of the bounds while meeting the diagonal dominance condition yielded satisfactory results: the magnitude of the control signals' oscillations was drastically reduced and, at the same time, we obtained adequate tracking and apparent robustness to the plant parameter change we investigated. We finally chose the bounds as $\overline{\beta}_{11} = \underline{\beta}_{11} = 1.2$, $\overline{\beta}_{22} = \underline{\beta}_{22} = 1.2$, and $\underline{\beta}_{21} = \underline{\beta}_{12} = 0.3$. Differentiation of the diagonal entries of $\mathbf{B}$ yields $M_{11} = 0.0$ and $M_{22} = 3.1\dot{\theta}_2$. We picked the fuzzy system approximation error bounds as $D_1 = 0.1$, $D_2 = 0.1$ (again, this is the result of a tuning process). The bounding functions $\overline{U}_1$ and $\overline{U}_2$ were picked in a way similar to Section IV-A, where the matrix coefficients are first chosen large and then decreased until adequate performance is achieved; setting the coefficients to 0.1 gave us the best results. Finally, the adaptation gains were set to $Q_{u_1} = Q_{u_2} = 4.71I$, where I is a 2×2 identity matrix.

We observe the control results on Fig. 2. We let $r_1(t)$ and $r_2(t)$ be square waves. Initially the controller has some difficulties and tracking is not perfect: at this point, the T–S coefficient matrices are moving away from zero and possibly adapting toward values that allow for better tracking. After the first period of the square wave reference inputs we note that tracking improves significantly. At time $t = 100$ s, when the system dynamics change as the robot "picks up" an object, we find the outputs exhibit virtually no transient overshoot, and tracking continues to be adequate. However, at this point we observe a high peak in τ_2 as the controller tries to compensate for the increase in the load of link two. The peaks reoccur at the transition points, where the references step up or down, but they tend to decrease in magnitude (we let the simulation run for 12 000 s, and found this pattern to hold).

V. Conclusions

In this paper, we have developed direct and indirect adaptive MIMO control schemes which use T–S fuzzy systems or a class of neural networks. We have proven stability of the methods and shown that they guarantee asymptotic convergence of the tracking errors to zero as well as boundedness of all the signals and parameter errors, regardless of any initialization constraints. Both methods allow for the inclusion of previous knowledge or expertise in form of linguistics regarding what the control input should be in the direct case or what the plant dynamics are, in the indirect case. We show that with or without such knowledge the stability and tracking properties of the controllers hold, and present two simulations for direct adaptive control that illustrate the method.

Acknowledgment

The authors would like to thank the reviewers for their encouraging comments and suggestions.

References

[1] T. Procyk and E. Mamdani, "A linguistic self-organizing process controller," *Automatica*, vol. 15, no. 1, pp. 15–30, 1979.

[2] D. Driankov, H. Hellendoorn, and M. M. Reinfrank, *An Introduction to Fuzzy Control.* Berlin, Germany: Springer-Verlag, 1993.

[3] J. R. Layne, K. M. Passino, and S. Yurkovich, "Fuzzy learning control for anti-skid braking systems," *IEEE Trans. Contr. Syst. Tech.*, vol. 1, pp. 122–129, June 1993.

[4] J. R. Layne and K. M. Passino, "Fuzzy model reference learning control for cargo ship steering," *IEEE Contr. Syst. Mag.*, vol. 13, pp. 23–34, Dec. 1993.

[5] W. A. Kwong, K. M. Passino, E. G. Lauknonen, and S. Yurkovich, "Expert supervision of fuzzy learning systems for fault tolerant aircraft control," *Proc. IEEE (Special Issue Fuzzy Logic Eng. Applicat.)*, vol. 83, pp. 466–483, Mar. 1995.

[6] V. G. Moudgal, W. A. Kwong, K. M. Passino, and S. Yurkovich, "Fuzzy learning control for a flexible-link robot," *IEEE Trans. Fuzzy Syst.*, vol. 3, pp. 199–210, May 1995.

[7] W. A. Kwong and K. M. Passino, "Dynamically focused fuzzy learning control," *IEEE Trans. Syst., Man, Cybern.*, vol. 26, pp. 53–74, Feb. 1996.

[8] J. T. Spooner and K. M. Passino, "Stable adaptive control using fuzzy systems and neural networks," *IEEE Trans. Fuzzy Syst.*, vol. 4, pp. 339–359, Aug. 1996.

[9] L.-X. Wang, *Adaptive Fuzzy Systems and Control: Design and Stability Analysis.* Englewood Cliffs, NJ: Prentice-Hall, 1994.

[10] ——, "Stable adaptive fuzzy control of nonlinear systems," in *Proc. 31st Conf. Decision Contr.*, Tucson, AZ, Dec. 1992, pp. 2511–2516.

[11] C.-Y. Su and Y. Stepanenko, "Adaptive control of a class of nonlinear systems with fuzzy logic," *IEEE Trans. Fuzzy Syst.*, vol. 2, pp. 285–294, Nov. 1994.

[12] T. A. Johansen, "Fuzzy model based control: Stability, robustness, and performance issues," *IEEE Trans. Fuzzy Syst.*, vol. 2, pp. 221–234, Aug. 1994.

[13] K. S. Narendra and K. Parthasarathy, "Identification and control of dynamical systems using neural networks," *IEEE Trans. Neural Networks*, vol. 1, pp. 4–27, Mar. 1990.

[14] M. M. Polycarpou and P. A. Ioannou, "Identification and control of nonlinear systems using neural network models: Design and stability analysis," Univ. Southern California, Elect. Eng. Syst. Rep. 91-09-01, Sept. 1991.

[15] R. M. Sanner and J.-J. E. Slotine, "Gaussian networks for direct adaptive control," *IEEE Trans. Neural Networks*, vol. 3, pp. 837–863, 1992.

[16] A. Yeşildirek and F. L. Lewis, "A neural network controller for feedback linearization," in *Proc. 33rd Conf. Decision Contr.*, Lake Buena Vista, FL, Dec. 1994, pp. 2494–2499.

[17] F.-C. Chen and H. K. Khalil, "Adaptive control of a class of nonlinear systems using neural networks," *Int. J. Contr.*, vol. 55, no. 6, pp. 1299–1317, 1992.

[18] G. A. Rovithakis and M. A. Christodoulou, "Adaptive control of unknown plants using dynamical neural networks," *IEEE Trans. Syst., Man, Cybern.*, vol. 24, pp. 400–412, Mar. 1994.

[19] C.-C. Liu and F.-C. Chen, "Adaptive control of nonlinear continuous-time systems using neural networks—General relative degree and MIMO cases," *Int. J. Contr.*, vol. 58, no. 2, pp. 317–355, 1993.

[20] F.-C. Chen and H. K. Khalil, "Adaptive control of linearizable discrete-time systems," in *34th IEEE Conf. Decision Contr. Proc.*, New Orleans, LA, Dec. 1995, pp. 2427–2432.

[21] R. Ordóñez and K. M. Passino, "Stable multi-input multi-output adaptive fuzzy/neural control," in *Proc. Amer. Contr. Conf.*, Albuquerque, NM, June 1997, pp. 1271–1272.

[22] R. E. Ordóñez, J. T. Spooner, and K. M. Passino, "Stable multi-input multi-output adaptive fuzzy control," in *IEEE Conf. Decision Contr.*, Kobe, Japan, Sept. 1996, pp. 610–615.

[23] S. S. Sastry and M. Bodson, *Adaptive Control: Stability, Convergence, and Robustness.* Englewood Cliffs, NJ: Prentice-Hall, 1989.

[24] S. S. Sastry and A. Isidori, "Adaptive control of linearizable systems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 1123–1131, Nov. 1989.

[25] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its application to modeling and control," *IEEE Trans. Syst., Man, Cybern.*, vol. 15, pp. 116–132, Jan. 1985.

[26] R. A. Horn and C. R. Johnson, *Matrix Analysis.* Cambridge (Cambridgeshire), NY: Cambridge Univ. Press, 1985.