## V. Conclusion

The technical note introduced a notion of linear i/o equivalence transformation for the set of meromorphic nonlinear higher order i/o difference equations. Then, it was proved that using the linear i/o equivalence transformations, the set of nonlinear equations can be transformed into the row-reduced form. Finally, the constructive algorithm is given for finding the equivalence transformation which extends the corresponding transformation for linear systems. The future task is to find out under which additional assumptions the concept of linear i/o equivalence coincides with the conventional i/o equivalence definition based on the i/o pairs.

The problem of transforming the set of i/o difference equations into a doubly-reduced (i.e., both row- and column-reduced) form is the topic of the future paper. For that purpose the paper [5] addressing the transformation the matrix over a skew polynomial ring into a doubly-reduced form, may be helpful. Note that the doubly-reduced form is instrumental in the solution of the realization problem of the i/o difference equations into the state space form.

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# Local Agent Requirements for Stable Emergent Group Distributions

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***Abstract*—This note introduces a model of a generic team formation problem. We derive general conditions under which a group of scattered decision-making agents converge to a particular distribution. The desired distribution is achieved when the agents divide themselves into a fixed number of sub-groups while consenting on "gains" which are associated to every subgroup. The model allows us to quantify the impact of limited sensing, motion, and communication capabilities on the rate at which the distribution is achieved. Finally, we show how this theory is useful in solving a cooperative surveillance problem.**

***Index Terms*—Cooperative systems, distributed decision-making, team formation.**

## I. Introduction

There is growing interest in designing and understanding multi-agent systems composed of independent decision-makers. Some well-known examples include cooperative groups of agents trying to accomplish a common *global* objective like: (*i*) agreeing upon a particular variable of interest (e.g., consensus problems [1], [2]); (*ii*) achieving collective group motion and formation patterns (e.g., [3], [4]); and (*iii*) allocating a group across spatially distributed tasks (e.g., [5]–[7]). While each of these problems has unique features, common constraints include (*i*) motion dynamics; (*ii*) range-limited and inaccurate sensing; or (*iii*) agent-to-agent limited and delayed communications.

In all of these multi-agent to multi-task studies there is a strong agent-to-task assignment coupling that occurs through the shared set of spatially distributed tasks that are to be accomplished by the group. For instance, suppose that tasks lie in distinct spatially separated areas. When a particular agent gets assigned to an area to perform tasks, the benefit of assigning all other agents to this area decreases since the same agent can usually perform several tasks in the same vicinity with

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little additional time. The challenge of how to untangle this complicated space and time dependent multi-agent to multi-task coupling to provide a desired cooperative group behavior – especially when the agents' motion, sensing, and communication constraints *dominate* the problem – remains an open problem, and is the key challenge we will address here.

Our work is closely related to [8]. There, the author introduces a model that captures the dynamics of a group of agents interacting via a network with time-dependent communication links. The general framework finds application in a variety of fields including swarming and consensus problems. Necessary and sufficient conditions are provided that guarantee that the individual agents' states (whose interpretation depends upon the particular application of the model) converge to a common value (actually, their results assume that for each agent the updated state is a strict convex combination of its own current state and the current states of the agents it connects to). As in [8] the stability analysis of the model presented in this note is based on upon a blend of graph and Lyapunov theory. However, as opposed to the time-driven model introduced in [8], here we assume that the evolution of the system is event-driven, and use a discrete event system (DES) modeling methodology similar to the one in [9].

The model we introduce in this note is built on a time-invariant graph with a finite number of nodes where agents may be located (i.e., a node refers to a sub-group of agents). Each node is associated with a unique "gain function" that characterizes the benefit of allocating a certain amount of agents to that node. We consider a general class of node gain functions which allows for diverse applications of the model, and focus on the individual agents' motion dynamics across the graph that lead to a *desired distribution* where the group as a whole tries to equalize the gains associated with any pair of nodes that are connected.

Our approach is inspired by techniques used in diffusion algorithms for load balancing (where loads move from heavily loaded processors to lightly loaded neighbor processors [10]–[12]). Although some ideas underlying our analysis are similar to consensus algorithms (like the one introduced in [8]), the convergence of diffusion algorithms cannot be derived from the corresponding results for consensus algorithms. In particular, diffusion algorithms do not require formation of convex combinations of the current or past states of the system. In this context, while the evolution inside a convex hull would allow consensus algorithms to ensure that over time the minimum and maximum gain levels of all nodes are monotonically increasing and decreasing, respectively, the convergence of our algorithm does not require that the minimum gain increases monotonically; and in fact this is not the case here. Our results are likely to be of interest in the area of distributed computing [13] since they extend the load balancing theory in [9], [12] to the continuous load case when the "virtual load" is a nonlinear function of the state.

It is, however, the general idea behind the model and strategies for achieving the desired distribution that is the most important contribution of this note. By achieving the desired distribution, we show a way to untangle the multi-agent to multi-task assignment coupling to provide good cooperative behavior, even from poorly informed and constrained individual decision-making. Similar coupling issues arise in a variety of allocation problems (i.e., where allocating a resource generally diminishes the benefit of allocating extra resources to that same site). A preliminary version of this work has appeared as a conference paper in [14].

## II. The Model

A common approach in modeling group behavior is to assume the existence of a large number of agents so that the total number of agents at a particular node can be adequately represented by a continuous variable. Here, we assume that there are $N \in \mathbb{N}$ nodes and let $x_i \in \mathbb{R}$, $x_i \geq \varepsilon_p$ represent the amount of agents at node $i$, $i = 1, 2, \ldots, N$, where $\varepsilon_p \geq 0$ is the minimum amount of agents allowed at any node. Each node is characterized by an associated gain function, defined as $s_i(x_i)$. We assume that for all $i = 1, \ldots, N$, and all $x_i \in [\varepsilon_p, P]$ the gain functions satisfy $s_i(x_i) > 0$; moreover, there exists two constants $a_i \geq b_i > 0$ such that

$$-a_i \leq \frac{s_i(y_i) - s_i(z_i)}{y_i - z_i} \leq -b_i \tag{1}$$

for any $y_i, z_i \in [\varepsilon_p, P]$, $y_i \neq z_i$. Equation (1) implies that the gain associated with each node decreases with an increasing amount of agents at that node, and eliminates the possibility that a very small difference in agents may result in an unbounded change in gain.

In order to model interconnections between nodes we will consider an undirected graph topology. The nodes are represented by $H = \{1, \ldots, N\}$ and the interconnection of nodes is described by $(H, A)$, where $A \subset H \times H$. If $(i, j) \in A$, this represents that an agent at node $i$ can sense node $j$ and can move from $i$ to $j$ (sensing node $j$ means that agents at node $i$ know $s_j(x_j)$ and $x_j$). Let $p(i) = \{j : \ (i, j) \in A\}$ represent all *neighboring* nodes of node $i$.

Next, let $\mathbb{R}_{\geq \varepsilon_p} = [\varepsilon_p, \infty)$ and $\Delta = \left\{x \in \mathbb{R}^N_{\geq \varepsilon_p} : \sum_{i=1}^N x_i = P\right\}$ be the simplex over which the $x_i$ dynamics evolve. Let $\mathcal{X} = \Delta$ be the set of states and $x(k) = [x_1(k), x_2(k), \ldots, x_N(k)]^\top \in \mathcal{X}$ be the state vector, with $x_i(k)$ representing the amount of agents at node $i$ at time index $k \geq 0$. We now want to define a set of states, such that any state $x(k)$ that belongs to this set exhibits the following desired characteristics:

— All neighboring nodes with more than $\varepsilon_p$ agents have equal gains;
— Any node that does not have the same gain as its neighboring nodes must have a lower gain and the minimum amount of agents $\varepsilon_p$ only.

Note that any distribution of agents such that the state belongs to the set

$$\begin{aligned}\mathcal{X}_c = \{x \in \mathcal{X} : \forall i \in H,\ either\ s_i(x_i) = s_j(x_j)\ \forall j \in p(i)\\ \text{such that } x_j \neq \varepsilon_p \text{ and } s_i(x_i) \geq s_j(x_j)\ \forall j \in p(i)\\ \text{such that } x_j = \varepsilon_p,\ or\ x_i = \varepsilon_p\}\end{aligned} \tag{2}$$

possesses the desired characteristics. In particular, any distribution $x \in \mathcal{X}_c$ is such that for any $i \in H$ either $x_i = \varepsilon_p$, in which case node $i$ has the minimum amount of agents allowed at that node (referred to as a *truncated* node); or if $x_i \neq \varepsilon_p$ it must be the case that all neighboring nodes $j \in p(i)$ such that $x_j \neq \varepsilon_p$ have the same gains as node $i$. The remainder of this section discusses the requirements which will ensure that (*i*) the set $\mathcal{X}_c$ is invariant under local interaction rules, and (*ii*) the group will converge to $\mathcal{X}_c$ from any initial distribution $x(0)$.

First, to capture the agent's dynamics at a particular node, let $e^{i,p(i)}_{\alpha(i)}$ denote an event of type $i$, which represents the movement of $\alpha(i)$ agents from node $i \in H$ to neighboring nodes $m \in p(i)$. The list $\alpha(i) = (\alpha_j(i), \alpha_{j'}(i), \ldots, \alpha_{j''}(i))$ where $j < j' < \cdots < j''$ and $j, j', \ldots, j'' \in p(i)$ is composed of elements $\alpha_m(i)$ that denote the amount of agents that move from node $i \in H$ to node $m \in p(i)$ (the size of the list $\alpha(i)$ is $|p(i)|$). For convenience, we will denote this list by $\alpha(i) = (\alpha_j(i) : \ \ j \in p(i))$. Let $\left\{e^{i,p(i)}_{\alpha(i)}\right\}$ denote the set of *all* possible combinations of how agents can move between nodes (i.e., $\alpha(i) \in \mathbb{R}^{|p(i)|}_{\leq P}$, where $\mathbb{R}_{\leq P} = [0, P]$). Let the set of events be described by $\mathcal{E} = \mathcal{P}\left(\left\{e^{i,p(i)}_{\alpha(i)}\right\}\right) - \{\emptyset\}$ ($\mathcal{P}(\cdot)$ denotes the power set). An event $e(k) \in \mathcal{E}$ is defined as a *set*, with each element of $e(k)$ representing the transition of possibly multiple agents among neighboring nodes in the graph $(H, A)$ (i.e., multiple elements in $e(k)$ represent the simultaneous movements of agents out of multiple nodes).

Second, to define the agents' sensing and motion conditions, we define an "enable function," $g : \mathcal{X} \longrightarrow \mathcal{P}(\mathcal{E}) - \{\emptyset\}$. In particular, if for a node $i \in H$, $s_i(x_i) \geq s_j(x_j)$ for all $j \in p(i)$, then $e_{\alpha(i)}^{i,p(i)} \in e(k)$ with $\alpha(i) = (0, \ldots, 0)$ is the only enabled event. Hence, agents at a node with the highest gain relative to all neighboring nodes do not move away from node $i$. On the other hand, if for node $i \in H$, $s_i(x_i) < s_j(x_j)$ for some $j$ such that $j \in p(i)$, then the only $e_{\alpha(i)}^{i,p(i)} \in e(k)$ are ones with $\alpha(i) = (\alpha_j(i) : j \in p(i))$ such that:

$$(i) \quad x_i - \sum_{m \in p(i)} \alpha_m(i) \geq \varepsilon_p,$$

$$(ii) \quad 0 < s_i\left(x_i - \sum_{m \in p(i)} \alpha_m(i)\right) \leq s_{j^*}\left(x_{j^*} + \alpha_{j^*}(i)\right)$$
$$\text{for some } j^* \in \{j : s_j(x_j) \geq s_m(x_m), \ \forall \ m \in p(i)\},$$

$$(iii) \; s_{j^*}\left(x_{j^*} + \alpha_{j^*}(i)\right) \leq s_{j^*}(x_{j^*}) - \gamma_{ij^*}(s_{j^*}(x_{j^*}) - s_i(x_i))$$
$$\text{for some } j^* \in \{j : s_j(x_j) \geq s_m(x_m), \ \forall \ m \in p(i)\}$$

where $\gamma_{ij} \in (0,1)$ for $j \in p(i)$ is a constant that represents the proportion of gain difference agents try to reduce by moving from node $i$ to node $j$. Condition (*i*) guarantees that at any node there are at least $\varepsilon_p$ agents. It is required so that conditions (*ii*) and (*iii*) are well defined at all times. To interpret conditions (*ii*) and (*iii*) it is useful to note that reducing (increasing) the amount of agents at a node always increases (decreases, respectively) the gain at that node. The two conditions constrain how agents can move in terms of node gains. Condition (*ii*) implies that *after* agents move from node $i$ to other nodes, the gain of node $i$ due to some agents leaving does not exceed at least one of the neighboring nodes with the highest gain (before agents started moving). Condition (ii) prevents there being too many agents moving from node i, so many that node i deliberately obtains a higher gain than all its neighbors. Without it, there could be a sustained movement oscillation between nodes. Condition (*iii*) implies that if the gain of node $i$ differs from any of its neighboring nodes, so that some agents may move from node $i$, then some agents must move to the node with highest gain. Without condition (*iii*) some high gain node could be ignored by the agents and the desired distribution might not be achievable. Condition (*ii*) together with condition (*iii*) guarantees that the highest gain node is strictly monotone decreasing over time as we show in the next section.

Third, we define a state transitions operator $f_e : \mathcal{X} \longrightarrow \mathcal{X}$, for $e(k) \in g(x(k))$. In particular, if $e(k) \in g(x(k))$ and $e_{\alpha(i)}^{i,p(i)} \in e(k)$, then $x(k+1) = f_{e(k)}(x(k))$, where $x_i(k+1)$ equals

$$x_i(k) - \sum_{\{j:\, j \in p(i),\, e_{\alpha(i)}^{i,p(i)} \in e(k)\}} \alpha_j(i) + \sum_{\{j:\, i \in p(j),\, e_{\alpha(j)}^{j,p(j)} \in e(k)\}} \alpha_i(j).$$

In other words, the amount of agents at node $i$ at time $k+1$, $x_i(k+1)$, is the amount of agents at node $i$ at time $k$, minus the total amount of agents leaving node $i$ at time $k$, plus the total amount of agents reaching node $i$ at time $k$.

Finally, to quantify the degree of asynchronism of the model, we assume that there exists a constant $B > 0$, such that in every substring $e(k'), e(k'+1), e(k'+2), \ldots, e(k'+(B-1))$ there is the occurrence of every type of event (i.e., for every $i \in H$, the event $e_{\alpha(i)}^{i,p(i)} \in e(k)$ for some $\alpha(i)$ that satisfies the above sensing and motion conditions at some time index $k$, $k' \leq k \leq k' + B - 1$). This assumption is met if agents try to move to neighboring nodes every certain number of steps. It bounds the slowest possible movement rate across nodes, and corresponds to the "partial asynchronism" assumption in [13].

## III. Stability of the Desired Distribution

We now restrict our mathematical analysis to the following scenario.

*Assumption 1:*
(a) The graph $(H, A)$ is connected;
(b) The gain functions $s_i$ associated to each node $i \in H$ satisfy (1);
(c) The amount of agents leaving each node $i \in H$ satisfy conditions (*i*) – (*iii*);
(d) The size of the group $P \geq P_c$ where $P_c = N\varepsilon_p + (N/\min_i\{b_i\})(\max_i\{s_i(\varepsilon_p)\} - \min_i\{s_i(\varepsilon_p)\})$.

In general, there are many different agent distributions such that $x \in \mathcal{X}_c$. Assumption 1 restricts our analysis to situations where there exists only one distribution that belongs to $\mathcal{X}_c$ and it has no truncated nodes. It requires that the size of the group exceeds a threshold $P_c$ and minimal restrictions on the graph $(H, A)$ (the particular value of $P_c$ depends on the given set of gain functions; for a more detailed discussion about the impact of group size on the desired distribution see [14]). Given Assumption 1 it may be possible in some special cases to explicitly find $x \in \mathcal{X}_c$. Our main result below (Theorem 3.4), however, is not dependent on knowing the explicit $x \in \mathcal{X}_c$. To establish Theorem 3.4, we present the following lemmas, the proofs of which are similar to the results in [7] and can be found in the supplement to this paper[1].

*Lemma 3.1:* Suppose a group of agents on a graph $(H, A)$ satisfy Assumption 1. Then, the set $\mathcal{X}_c$ is unique and invariant.

Since the set $\mathcal{X}_c$ is unique (i.e., $|\mathcal{X}_c| = 1$), for any initial agent distribution $x(0)$ there exists only one distribution that represents the desired distribution. To study how the group of agents approach this distribution, we build on the following two lemmas.

*Lemma 3.2:* Suppose a group of agents on a graph $(H, A)$ satisfy Assumption 1. Then, for all $i \in H$

$$s_i(x_i(k+1)) \leq \max_i\{s_i(x_i(k))\} - \gamma[\max_i\{s_i(x_i(k)\} - s_i(x_i(k))] \quad (3)$$

where $\gamma = \min_{ij}\{\gamma_{ij}\}$.

Note that (3) applies to all $i \in H$ and in particular to some $i$ such that $s_i(x_i(k)) \geq s_j(x_j(k))$ for all $j \in H$. Hence, $\max_i\{s_i(x_i(k))\}$ (i.e., the highest gain in the entire the graph) is a nonincreasing function of $k$. We use Lemma 3.2 to bound the gain levels of the neighboring nodes of any node $i \in H$ with more than $\varepsilon_p$ agents.

*Lemma 3.3:* Suppose a group of agents on a graph $(H, A)$ satisfy Assumption 1. Then, for all $i \in H$ and $j \in p(i)$ such that $x_i(k) \neq \varepsilon_p$

$$s_j(x_j(k')) \leq \max_i\{s_i(x_i(k))\} - \gamma^{k'-k}[\max_i\{s_i(x_i(k))\} - s_i(x_i(k))] \quad (4)$$

for all $k' \geq k + NB$.

Finally, we are ready to present our main result. The proof of the following theorem is presented in the Appendix.

*Theorem 3.4:* Suppose a group of agents on a graph $(H, A)$ satisfy Assumption 1. Then, the invariant set $\mathcal{X}_c$ is exponentially stable in the large.

Exponential stability of an invariant set means that all agents are guaranteed to converge to $\mathcal{X}_c$ at a certain rate. Theorem 3.4 is an extension of the load balancing [13] theorems in [9], [12] to the case when the "virtual load" is a nonlinear function of the state. In particular, the virtual case in [12] represents the case when $s_i(x_i) = T - \beta_i x_i$, where $T = P\max_i\{\beta_i : i \in H\}$ is some positive constant that guarantees that $s_i(x_i) > 0$ for all $x_i$. The authors in [12] consider load processors that may process load at different rates. It is then useful to scale the physical load by assigning constants $\beta_i > 0$ which are inversely proportional to the processing rate of node $i$. Note that $s_i$ represents a line with negative slope which passes through the origin. Faster processing processors have a lower $\beta_i$ value, which corresponds to a line

[1]available at www.ece.osu.edu/~passino/kmp-pubs.html.

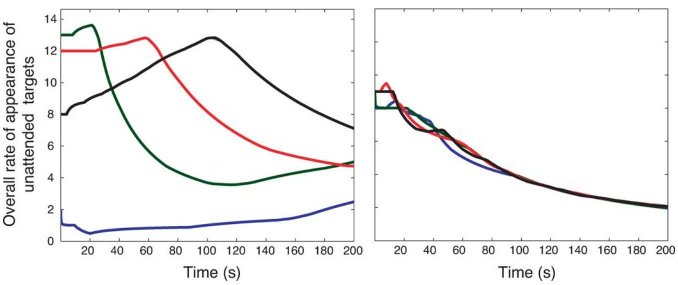


Fig. 1. Gains for all four areas; greedy strategy (left), cooperative strategy (right).

with a lower slope. If one then draws a horizontal line representing the desired distribution, it would cross lines with steeper slopes closer to the $y$-axis than lines with lower slopes. Thus, faster processors receive more load than slower processors as expected.

## IV. Application

Here, we assume that there are $P = 16\ \mathrm{discrete\ agents}$ which must cover a region defined by a square area. We assume that the region is divided into four equally sized areas. Every node $i \in H$ represents an area. Let us assume that every four seconds a new pop up target randomly appears anywhere in the region. We also assume that agents can move from any area to any other area and consider therefore a complete graph $(H, A)$. Moreover, agents have complete information about the region and may even share information with agents that are not necessarily in the same area. In particular, we assume that the location of targets which have appeared in the region is known to every agent (e.g., via satellite information). We let $\varepsilon_p = 0$ so that some areas may have no agents in them at all. If an agent approaches a target within a area, the target is considered to be "attended." Once the target is reached, the agent may perform tasks such as classification, engagement, or verification of the target, and it is then ignored for the rest of the mission. Gain functions for every area are defined as the overall rate of appearance of unattended targets. In particular, the gain of node $i$ is defined as the number of targets present in that area that are not being or have not been visited by any agent in a time window divided by the length of that window (hence this is an approximation of the overall rate of appearance of unattended targets). If an agent approaches a target located in area $i$, this will decrease the gain in that area and increase the overall target service rate. Our goal is to achieve similar overall target service rates in all areas.

In order to evaluate different agent strategies, we define the mission performance measure as the time needed for the difference between any two gains to reach and settle within a given range (here 4%). Note that since our simulations assume discrete agents, perfect consensus of all gains cannot be achieved in general. We denote the settling time for a given mission by $t_s$. We will first compare the performance of the proposed agent strategy to a "greedy" strategy, where every time agents reach a target they successively decide to approach the area with the highest gain. The left plot in Fig. 1 shows how gains change over time during the first 200 s of a mission. It represents the case when agents violate the proposed conditions (*i*) – (*iii*), and simply approach the area with the highest rate of appearance of unattended targets. Note that the gains do not converge to any particular value. On the other hand, the right side plot in Fig. 1 represents the case when agents distribute themselves over the region while satisfying conditions (*i*) – (*iii*). Here, we assume that $\gamma_{ij} = 0.01$ for all $(i, j) \in A$. Note that the gains converge and the settling time in this case is approximately $t_s = 100$ s. Moreover, for any time $t \geq t_s$ the overall rate of appearance of unattended targets in all areas differs by less than 4%.

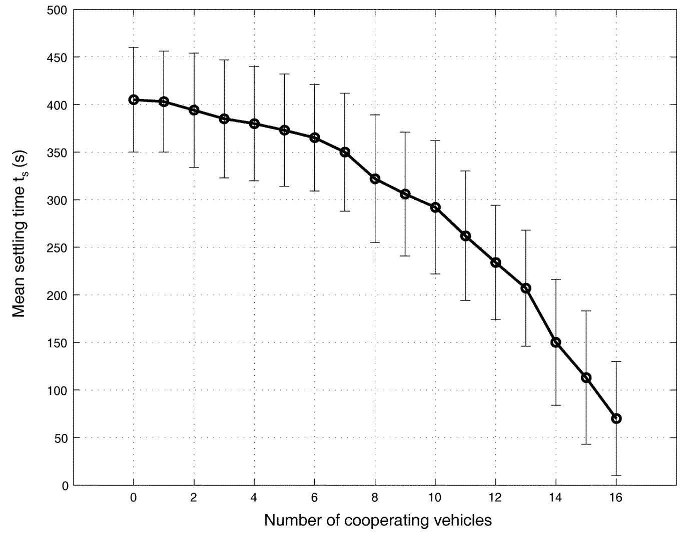


Fig. 2. Settling time for different cooperation levels. Every data point represents 60 simulation runs with varying target pop up locations. The error bars are sample standard deviations for these runs.

Next, we want to show the effect of different cooperation levels between agents and how increasing communication between them affects the settling time of the system. Fig. 2 shows how the average settling time decreases with the number of agents any agent may cooperate with (i.e., by sharing information on where to go). Note that sharing information between two agents on where to go only becomes useful if it affects another agent's perception about the gain of an area, and consequently its decision on what target to pursue. Therefore, several factors such as the total number of agents, the rate of appearance of targets, the agents' speed, and the simulation step time shape the curve in Fig. 2. For example, the more agents are present, the higher the probability that simultaneous decisions are made, and the more beneficial it becomes to share information. Increasing the number of agents lowers the settling time and the standard deviations of the runs.

## V. Conclusion

We introduce a mathematical model that captures essential aspects that arise due to the coupling between the assignment of spatially distributed tasks to all cooperating agents. In particular, the proposed model takes into account that when an agent is assigned to a particular area to perform certain tasks, the benefit of assigning this area to all other agents decreases, and suggests a way to untangle this space and time dependent coupling and better exploit the benefits of cooperation.

While most of the current research on cooperative decision-making focuses on groups of identical agents, developing formal frameworks that allow us to consider heterogeneous agents remains a future research direction. Such agent diversity is an essential feature that both demands and exploits cooperation.

## Appendix
## Proof of Theorem 3.4

Let $x' = [x'_1, \ldots, x'_N]^\top$ and choose

$$\rho(x, \mathcal{X}_c) = \inf\{\max_i\{|x_i - x'_i| : \ i \in H\} : x' \in \mathcal{X}_c\} \tag{5}$$

$$V(x) = \max_i\{s_i(x_i)\} - \frac{1}{N}\sum_{j\in H} s_j(x_j). \tag{6}$$

Note that for $x \in \mathcal{X}_c$, $V(x) = 0$ since all nodes $i \in H$ have the same gain.

To show that $V(x)$ is underbounded by a class $K$ function $c_1\rho(x, \mathcal{X}_c)$, note that according to (1) for all $x_i \in [\varepsilon_p, P]$ and all $i \in H$, it must be the case that for any $x \notin \mathcal{X}_c$ and $x' \in \mathcal{X}_c$, there is some node $i \in H$ such that $x_i \neq x'_i$ and two constants $c_1 > 0$ and $\underline{b} = \min_i\{b_i\}$ such that

$$\frac{s_i(x_i) - s_i(x'_i)}{x_i - x'_i} \leq -b_i \leq -\underline{b} < -\frac{b}{N} = -c_1 < 0. \quad (7)$$

Since (7) applies for any $i \in H$ such that $x_i \neq x'_i$, it must apply for some node

$$j = \arg\max\{|x_i - x'_i| : \ i \in H, \ x_i \neq x'_i\}. \quad (8)$$

Assume $j$ is this value for the next part of the proof. Using the definition of $\rho(x, \mathcal{X}_c)$ and applying (7) to node $j$ yields

$$\begin{aligned}\underline{b}\rho(x, \mathcal{X}_c) &\leq \underline{b}\max\{|x_i - x'_i| : i \in H, \ x_i \neq x'_i\} \\ &= \underline{b}|x_j - x'_j| \leq |s_j(x_j) - s_j(x'_j)|. \quad (9)\end{aligned}$$

Note that for any agent distribution $x \notin \mathcal{X}_c$ and $x' \in \mathcal{X}_c$, one of the following must be true: In the first case, if on the right side of (9) $s_j(x_j) - s_j(x'_j) > 0$, that is, if node $j$ needs to gain some agents to achieve the desired state, then there *must* exist some other node $j^*$ such that

$$s_{j^*}(x_{j^*}) - s_{j^*}(x'_{j^*}) < 0. \quad (10)$$

In other words, there must exist another node $j^*$ that needs to loose some agents to achieve *its* desired state $x'_{j^*}$ (e.g., some node where some of the agents needed at node $j$ could come from). Hence, $x_{j^*} \neq \varepsilon_p$.

Moreover, since there are no truncated nodes at the desired distribution, $x'_{j^*} \neq \varepsilon_p$. Hence, $s_{j^*}(x'_{j^*}) = s_j(x'_j)$. Then according to (10), $s_j(x'_j) > s_{j^*}(x_{j^*})$. Therefore

$$\begin{aligned}0 &< s_j(x_j) - s_j(x'_j) < s_j(x_j) - s_{j^*}(x_{j^*}) \\ &\leq \max_i\{s_i(x_i)\} - \min_i\{s_i(x_i) : x_i \neq \varepsilon_p\}.\end{aligned}$$

In the second case for (9), if $s_j(x_j) - s_j(x'_j) < 0$, that is, if node $j$ needs to loose some agents to achieve the desired state (so $x_j \neq \varepsilon_p$), then there must also exist some other node $j^*$ such that $s_{j^*}(x_{j^*}) - s_{j^*}(x'_{j^*}) > 0$. In other words, there must exist another node $j^*$ that needs to gain some agents to achieve its desired state $x'_{j^*}$. Again, $x'_{j^*} \neq \varepsilon_p$ since there are no truncated nodes at the desired distribution, and so $s_{j^*}(x'_{j^*}) = s_j(x'_j)$. Hence

$$\begin{aligned}0 &< s_j(x'_j) - s_j(x_j) = s_{j^*}(x'_{j^*}) - s_j(x_j) \leq s_{j^*}(x_{j^*}) - s_j(x_j) \\ &\leq \max_i\{s_i(x_i)\} - \min_i\{s_i(x_i) : x_i \neq \varepsilon_p\}.\end{aligned}$$

Thus, (9) can be overbounded by $\max_i\{s_i(x_i)\} - \min_i\{s_i(x_i) : x_i \neq \varepsilon_p\}$ so that

$$\begin{aligned}\underline{b}\rho(x, \mathcal{X}_c) &\leq \max_i\{s_i(x_i)\} - \min_i\{s_i(x_i) : x_i \neq \varepsilon_p\} \\ &\leq \max_i\{s_i(x_i)\} - \min_i\{s_i(x_i)\}. \quad (11)\end{aligned}$$

From now on we do not necessarily assume that $j$ is defined via (8).

Next, note that

$$\begin{aligned}&V(x) \\ &\geq \max_i\{s_i(x_i)\} - \frac{1}{N}\left[\min_i\{s_i(x_i)\} + (N-1)\max_i\{s_i(x_i)\}\right] \\ &\geq \frac{1}{N}\left[\max_i\{s_i(x_i)\} - \min_i\{s_i(x_i)\}\right].\end{aligned}$$

Using (11) we get

$$\frac{b}{N}\rho(x, \mathcal{X}_c) \leq \frac{1}{N}\left[\max_i\{s_i(x_i)\} - \min_i\{s_i(x_i)\}\right] \leq V(x). \quad (12)$$

Thus, $V(x) \geq c_1\rho(x, \mathcal{X}_c)$ for all $x \in \mathcal{X}$.

Next, we will show that there exists a constant $c_2$ such that $V(x) \leq c_2\rho(x, \mathcal{X})$ for all $x \in \mathcal{X}$. Let $\overline{a} = \max_i\{a_i\}$. Recall that for all $x \notin \mathcal{X}_c$ and $x' \in \mathcal{X}_c$, $\max\{|x_i - x'_i| : i \in H\} > 0$. Note also that if $j = \arg\max_i\{s_i(x_i)\}$, then according to (1)

$$0 \leq \left|\frac{\max_i\{s_i(x_i)\} - s_j(x'_j)}{\max\{|x_i - x'_i| : i \in H\}}\right| \leq \overline{a} \quad (13)$$

and similarly, if $m = \arg\min_i\{s_i(x_i)\}$, then

$$0 \leq \left|\frac{s_m(x'_m) - \min_i\{s_i(x_i)\}}{\max\{|x_i - x'_i| : i \in H\}}\right| \leq \overline{a}. \quad (14)$$

By adding (13) and (14) get that

$$\begin{aligned}2\overline{a} &\geq \frac{|\max_i\{s_i(x_i)\} - s_j(x'_j)| + |s_m(x'_m) - \min_i\{s_i(x_i)\}|}{\max\{|x_i - x'_i| : i \in H\}} \\ &\geq \frac{|\max_i\{s_i(x_i)\} - s_j(x'_j) + s_m(x'_m) - \min_i\{s_i(x_i)\}|}{\max\{|x_i - x'_i| : i \in H\}}.\end{aligned}$$

Note that $s_m(x'_m) - s_j(x'_j) = 0$, since $x' \in \mathcal{X}_c$ and there are no truncated nodes at the desired distribution. Thus

$$\frac{\max_i\{s_i(x_i)\} - \min_i\{s_i(x_i)\}}{\max\{|x_i - x'_i| : i \in H\}} \leq 2\overline{a}.$$

Moreover

$$\begin{aligned}V(x) &\leq \max_i\{s_i(x_i)\} - \frac{1}{N}(N\min_i\{s_i(x_i)\}) \\ &\leq \max_i\{s_i(x_i)\} - \min_i\{s_i(x_i)\}.\end{aligned}$$

Hence, if $c_2 = 2\overline{a}$, $V(x) \leq c_2\max\{|x_i - x'_i| : \ i \in H\}$. This bound applies to any $x' \in \mathcal{X}_c$ and $x \notin \mathcal{X}_c$ and according to the definition of $\rho(x, \mathcal{X}_c)$

$$V(x) \leq c_2\inf\{\max_i\{|x_i - x'_i| : i \in H\} : x' \in \mathcal{X}_c\} = c_2\rho(x, \mathcal{X}_c). \quad (15)$$

Thus, $V(x) \leq c_2\rho(x, \mathcal{X}_c)$ for all $x \in \mathcal{X}$.

Next, we show that the distance from the trajectories to the $\mathcal{X}_c$ is overbounded by an exponentially decreasing function of $k$. To do so, we first extend the result presented in Lemma 3.3, that is, (4) to

$$\begin{aligned}&s_j(x_j(k')) \leq \max_i\{s_i(x_i(k))\} \\ &-(\gamma^{k'-k})^l\left[\max_i\{s_i(x_i(k))\} - s_i(x_i(k))\right] \text{for all } k' \geq k + lNB \quad (16)\end{aligned}$$

where $j$ is any node that is reachable from $i$ by spanning $l$ inter-node connections (arcs $(i, j) \in A$). Equation (4) establishes the validity of (16) for $l = 1$. We assume (16) is valid for a general $j$ at a distance $l$ from $i$, such that exists some node $q \in p(j)$, such that $q$ is at a distance $l + 1$ from $i$. Equation (4), with $k$ replaced by $N + lNB$ and therefore

$k' \geq (k + lNB) + NB$, and applied to a neighboring node of node $j$, $q \in p(j)$, gives that for all $k' \geq k + (l+1)NB$

$$\begin{aligned} & s_q(x_q(k')) \\ & \leq \max_i\{s_i(x_i(k+lNB))\} \\ & \quad -\gamma^{k'-(k+lNB)}\left[\max_i\{s_i(x_i(k+lNB))\} - s_j(x_j(k+lNB))\right] \\ & \leq \max_i\{s_i(x_i(k))\} \\ & \quad - \gamma^{k'-k}\left[\max_i\{s_i(x_i(k))\} - s_j(x_j(k+lNB))\right]. \end{aligned}$$

Substituting, based on our inductive hypothesis, it follows that for all $k' \geq k + (l+1)NB$

$$\begin{aligned} & s_q(x_q(k')) \\ & \leq \max_i\{s_i(x_i(k))\} \\ & \quad - \gamma^{k'-k}\left[\max_i\{s_i(x_i(k))\} - \left[\max_i\{s_i(x_i(k))\}\right.\right. \\ & \quad \left.\left. -(\gamma^{k'-k})^l\left[\max_i\{s_i(x_i(k))\} - s_i(x_i(k))\right]\right]\right] \\ & = \max_i\{s_i(x_i(k)\} \\ & \quad - (\gamma^{k'-k})^{l+1}\left[\max_i\{s_i(x_i(k))\} - s_i(x_i(k))\right]. \end{aligned}$$

Hence, (16) must be valid for all $l \geq 1$.

Because every node in the graph can be reached from $i$ by spanning fewer than $N$ arcs, (16) implies that

$$\begin{aligned} s_j(x_j(k')) \leq & \max_i\{s_i(x_i(k))\} \\ & -(\gamma^{k'-k})^N\left[\max_i\{s_i(x_i(k))\} - s_i(x_i(k))\right] \end{aligned} \quad (17)$$

for all $k' \geq k + N^2B$, $j \in H$. Because we have made no further assumptions, (17) is valid for any $i \in H$ such that $x_i(k) \neq \varepsilon_p$. Hence, we can replace $s_i(x_i(k))$ with $\min_i\{s_i(x_i(k)) : x_i(k) \neq \varepsilon_p\}$ and (17) becomes

$$\begin{aligned} s_j(x_j(k')) \leq & \max_i\{s_i(x_i(k))\} \\ & -(\gamma^{k'-k})^N\left[\max_i\{s_i(x_i(k))\} - \min_i\{s_i(x_i(k)) : x_i(k) \neq \varepsilon_p\}\right] \end{aligned}$$

for all $k' \geq k + N^2B$, $j \in H$. It follows directly that for all $k' \geq k + N^2B$

$$\begin{aligned} & \max_i\{s_i(x_i(k'))\} \leq \max_i\{s_i(x_i(k))\} \\ & - (\gamma^{k'-k})^N\left[\max_i\{s_i(x_i(k))\} - \min_i\{s_i(x_i(k)) : x_i(k) \neq \varepsilon_p\}\right]. \end{aligned} \quad (18)$$

Choose $k' = k + N^2B$. For every $k \geq 0$, $x(k) \notin \mathcal{X}_c$, since there are no truncated nodes at the desired distribution, (11) and (18) imply that

$$\begin{aligned} & \max_i\{s_i(x_i(k))\} - \max_i\{s_i(x_i(k+N^2B))\} \\ & \geq (\gamma^{N^2B})^N\left[\max_i\{s_i(x_i(k))\} - \min_i\{s_i(x_i(k)) : x_i(k) \neq \varepsilon_p\}\right] \\ & \geq \gamma^{N^3B}\underline{b}\rho(x(k), \mathcal{X}_c).. \end{aligned} \quad (19)$$

Equation (19) together with the lower and upper bound on $V(x(k))$ satisfy sufficient conditions for exponential stability of the invariant set $\mathcal{X}_c$ [9].


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