Stable Auto-Tuning of Adaptive Fuzzy/Neural Controllers for Nonlinear Discrete-Time Systems

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***Abstract*—In direct adaptive control, the adaptation mechanism attempts to adjust a parameterized nonlinear controller to approximate an ideal controller. In the indirect case, however, we approximate parts of the plant dynamics that are used by a feedback controller to cancel the system nonlinearities. In both cases, "approximators" such as linear mappings, polynomials, fuzzy systems, or neural networks can be used as either the parameterized nonlinear controller or identifier model. In this paper, we present algorithms to tune some of the parameters (e.g., the adaptation gain and the direction of descent) for a gradient-based approximator parameter update law used for a class of nonlinear discrete-time systems in both direct and indirect cases. In our proposed algorithms, the adaptation gain and the direction of descent are obtained by minimizing the instantaneous control energy. We will show that updating the adaptation gain can be viewed as a special case of updating the direction of descent. We will also compare the direct and indirect adaptive control schemes and illustrate their performance via a simple surge tank example.**

***Index Terms*—Adaptive control, fuzzy/neural control.**

I. Introduction

CONSIDER the discrete-time single-input–single-output (SISO) nonlinear system described by

$$y(k+d) = f\left(x(k), u(k)\right) \tag{1}$$

where $f(\cdot)$ is a smooth function, $x(k)$ is a state vector, u is a scalar input, y is the scalar output, and $d \geq 1$. We will consider cases where there is significant uncertainty in our knowledge of $f(\cdot)$, and we will consider several adaptive control approaches to cope with this problem. In adaptive control, typically some type of function approximator is used to approximate the plant dynamics (for the indirect approach) or the controller dynamics (in the direct case). Good candidates for such approximators (that include linear mappings, polynomials, wavelets, and many others) are fuzzy systems and neural networks that are known to posses the universal approximation property [1]–[4].

Many techniques for adaptive control of discrete-time nonlinear systems have been developed. First, we outline some of the conventional control approaches for discrete-time nonlinear systems that include adaptive output feedback control as in [5] and [6]. In [7], the authors consider H_∞-control of discrete time nonlinear systems. They studied the problem of disturbance attenuation under different assumptions. An adaptive control scheme for nonlinear systems (described by the scalar system $y(k+1) = a^*\zeta(y(k)) + u(k)$ where a^* is an unknown constant and ζ is a nonlinear function which is not necessarily sector bounded) is presented in [8] and [9]. In [10], necessary and sufficient conditions for the equivalence between a general class of discrete-time systems and discrete-time systems in strict feedback form were provided using the backstepping method.

The authors in [11] presented an indirect adaptive control scheme using neural networks where stability is ensured assuming that the neural networks are poorly adjusted. In [12], however, the authors ensured tracking to an ϵ-neighborhood of zero when they relaxed the above assumption. Stability was ensured in [11] and [12], assuming that the initial estimate of the system is close to the actual one. In [13] and [14], the authors consider a discrete time multiple-input–multiple-output nonsquare feedback linearizable system without zero dynamics, but with a bounded state disturbance. They designed an indirect adaptive controller using neural networks (linear in the parameters in [13], and nonlinear in the parameters in [14]) which require no initialization conditions. Such a controller provides uniform boundedness of the tracking error. In [15], a discrete-time fuzzy logic controller for a class of unknown feedback linearizable nonlinear dynamical systems is presented. Unlike most adaptive control approaches, this fuzzy controller uses basis functions based on the fuzzy system, not a regression matrix. The significance of this approach is that no certainty equivalence assumption is needed and no assumption of linearity in the parameters is used. In [16] the author considers a NARX discrete time system with unknown uncertainties for which a dynamical upper bound is known. An adaptive controller in the form of a "parallel distributed compensator" is designed. This controller uses a Takagi–Sugeno fuzzy system (TSFS) that is updated online. Finally, the authors in [17] and [18] considered a class of discrete-time nonlinear systems which includes strict feedback systems. In [18], a direct adaptive control scheme is presented where TSFS were used as a functional approximator. A continuous dead zone is used to guarantee convergence of the output tracking error to an ϵ-neighborhood of zero. In [17] the authors presented an indirect adaptive control scheme using TSFS. Similar stability results are achieved using a continuous dead zone (in which a gradient method is used for adaptation) and discontinuous dead zone (in which least squares method is used for adaptation). It is important to mention that the work presented in [12], [17], and [18] is the most relevant work to ours; actually, some of the notations and definitions we use here are exactly the same.

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Within this study, we consider a feedback linearizable subclass of discrete-time nonlinear systems described by (1). Similar to [12], [17], and [18], the parameter vectors of the approximators are updated using a gradient update law. Unlike [12], [17], and [18] (where the adaptation gain and direction of descent are not updated), the adaptation gain and direction of descent used here are updated in ways that seek to optimize certain cost functions. For both the direct and indirect cases, we extend the stability analysis in [12] to prove that the adaptation gain and direction of descent tuning methods provide stable operation. Moreover, we provide a surge tank example that illustrates the ideas presented here.

II. Direct and Indirect Adaptive Control

In this section, we start by describing the system we consider for control, along with its direct and indirect control laws, and adaptation law.

A. Plant Description and Control Laws

Here, we consider the SISO discrete-time system described by (as shown in [12])

$$y(k+1) = f_0\left(x(k)\right) + g_0\left(x(k)\right) u(k-d+1) \tag{2}$$

where $f_0(\cdot)$ and $g_0(\cdot)$ are unknown smooth functions, $x(k)$ is a vector of past inputs and outputs $[y(k-n+1), \ldots, y(k), u(k-m-d+1), \ldots, u(k-d)]^\top$, where $m \leq n$, y is the output, u is the input, and d is the time delay (relative degree) of the system. It is known that for the class of systems (2), there exists an ideal controller $(u^*(k))$ that drives the output of the system to track a known reference trajectory after d steps. Such a controller is defined as

$$u^*(k) = \frac{-f_{d-1}\left(x(k)\right) + r(k)}{g_{d-1}\left(x(k)\right)} \tag{3}$$

where it can be shown by recursive substitution as in [12] that $f_{d-1}(x(k)) = f_0(x(k+d-1))$ and $g_{d-1}(x(k)) = g_0(x(k+d-1))$. Here, we consider the same plant assumptions used in [12].

A direct adaptive controller that seeks to drive the system to track a known reference input $r(k)$ uses an approximator that attempts to approximate the ideal controller dynamics (u^*, that we assume to exist). Here, we assume that the ideal control can be approximated by

$$u(k) = A_u^\top \zeta\left(x(k), r(k)\right) + u_k(k) \tag{4}$$

where $A_u(k)$ is an approximation of the ideal parameter vector A_u^*, $u_k(k)$ is the known part of the ideal control, and $\zeta(x, r)$ is the partial of the approximator output with respect to the parameter vector. Define the approximator parameter error as $\phi(k) = A_u(k) - A_u^*$.

Unlike the direct approach, in the indirect approach we approximate the plant dynamics, then the feedback controller uses these estimates of the plant dynamics. As in the direct case, let us consider the subclass of systems (2) which can be written as

$$\begin{aligned} y(k+d) &= f_{d-1}\left(x(k)\right) + g_{d-1}\left(x(k)\right) u(k) \\ &= f_u\left(x(k)\right) + f_k\left(x(k)\right) \\ &\quad + \left[g_u\left(x(k)\right) + g_k\left(x(k)\right)\right] u(k) \end{aligned} \tag{5}$$

where $f_k(\cdot)$ and $g_k(\cdot)$ are the known parts of the dynamics, and $f_u(\cdot)$ and $g_u(\cdot)$ are the unknown parts of the dynamics (in what follows, we can consider the case where $f_k \equiv g_k \equiv 0$).

Using the certainty equivalence approach, the control law is defined as

$$u(k) = \frac{-\hat{f}_{d-1}\left(x(k)\right) + r(k)}{\hat{g}_{d-1}\left(x(k)\right)} \tag{6}$$

where $\hat{f}_{d-1}(x(k))$ and $\hat{g}_{d-1}(x(k))$ are estimates of $f_{d-1}(x(k))$ and $g_{d-1}(x(k))$, respectively. A projection algorithm may be used to ensure that $\hat{g}_{d-1}(x(k)) \geq \theta_0 > 0$ so that the control signal is well defined. As in the direct case, the parameter errors for the indirect adaptive controller are defined as $\phi_f(k) = A_f(k) - A_f^*$ and $\phi_g(k) = A_g(k) - A_g^*$.

Note that fuzzy systems can be used as approximators to approximate the controller in the direct case, or approximate parts of the plant dynamics as in the indirect case. One good candidate of fuzzy systems is the TSFS, which has shown to be successful in many applications.

The error equation for both direct and indirect cases can be written as

$$e(k+1) = -\kappa\theta\left(x(k-d+1)\right)\phi^\top(k)\zeta\left(x(k-d+1)\right) + \bar{\nu}(k) \tag{7}$$

where $\theta(x(k)) = g_0(x(k+d-1))$ (it is assumed here that $\theta(x(k))$ is defined such that $0 < \theta_0 \leq \theta(x(k)) \leq \theta_1$, and θ_0 and θ_1 are known constants related to the plant dynamics), $\theta(x(k-d+1)) = 1$ in the indirect case, and $\kappa = 1$ and -1 in the direct and indirect cases, respectively. Also, $\bar{\nu}(k)$ is function of the approximation error. For simplicity, we will write (7) as

$$e(k+1) = -\kappa\theta(k-d+1)\phi^\top(k)\zeta(k-d+1) + \bar{\nu}(k). \tag{8}$$

Here, the normalized gradient-based parameter update law (that seeks to minimize the squared tracking error) is used. Consider the cost function

$$J(A, \eta) = J_e(A) + J_u(\eta) \tag{9}$$

where $J_e(A) = e^2(k)$, $J_u(\eta) = u^2(k)$, $e(k)$ is the instantaneous tracking error, and $u(k)$ is the instantaneous control, and η (as we will define later) is the adaptation gain.

The normalized gradient-based parameter update law that minimizes the first part of the cost function (i.e., $J_e(A)$) can be expressed as

$$A(k) = A(k-1) + \frac{\kappa\eta\zeta(k-d)}{1 + \gamma\left|\zeta(k-d)\right|^2} e_\epsilon(k) \tag{10}$$

where η is the adaptation gain, and γ is a positive constant ($\gamma > 0$). Here, $e_\epsilon(k)$ (which is the representation of the output error $e(k)$ in terms of a continuous dead zone of finite size $\epsilon > 0$) is defined as

$$e_\epsilon\left(e(k),\epsilon\right) = \begin{cases} e(k)-\epsilon & \text{if } e(k) > \epsilon \\ 0 & \text{if } |e(k)| \le \epsilon \\ e(k)+\epsilon & \text{if } e(k) < -\epsilon \end{cases}. \quad (11)$$

It is assumed that η is chosen such that

$$0 < \eta < \frac{2\gamma}{\theta_1}. \quad (12)$$

It can shown that, $e_\epsilon(k)$ can be written as

$$e_\epsilon(k) = -\kappa\pi(k)\theta(k-d)\phi^\top(k-d)\zeta(k-d) \quad (13)$$

where $0 \le \pi(k) < 1$.

B. Stability Analysis

In this section, we state stability and convergence results for the system (2) (for both direct and indirect cases) similar to the ones presented in [12] for the indirect case.

Theorem 1: Suppose $|r(k)| \le d_1$ for all $k \ge 0$. Given any constant $\varrho > 0$ and any small constant $\epsilon > 0$, there exist positive constants $\varrho_1 = \varrho_1(\varrho, d_1)$, $\varrho_2 = \varrho_2(\varrho, d_1)$, $\varepsilon^* = \varepsilon^*(\varrho, \epsilon, d_1)$, and $\delta^* = \delta^*(\varrho, \epsilon, d_1)$ such that if the appropriate assumptions stated in [12] are satisfied on $\mathcal{S}_x \supset B_{\varrho_1}$ with $\varepsilon < \varepsilon^*$, and also satisfied on B_{ϱ_2}, $|x(0)| \le \varrho$, and $|\phi(0)| \le \delta < \delta^*$, then using the direct adaptive control law (4), we will ensure that

1) $|\phi(k)|$ will be monotonically nonincreasing, and $|\phi(k) - \phi(k-1)|$ will converge to zero;
2) the tracking error between the plant output and the reference command will converge to a ball of radius ϵ centered at the origin.

Proof: This proof follows the proof of the indirect case presented in [12]. ∎

The authors in [12] have derived local convergence results for indirect adaptive control as shown in next theorem.

Theorem 2: Suppose $|r(k)| \le d_1$ for all $k \ge 0$. Given any constant $\varrho > 0$ and any small constant $\epsilon > 0$, there exist positive constants $\varrho_1 = \varrho_1(\varrho, d_1)$, $\varrho_2 = \varrho_2(\varrho, d_1)$, $\varepsilon^* = \varepsilon^*(\varrho, \epsilon, d_1)$, and $\delta^* = \delta^*(\varrho, \epsilon, d_1)$ such that if the appropriate sssumptions stated in [12] are satisfied on $\mathcal{S}_x \supset B_{\varrho_1}$ with $\varepsilon < \varepsilon^*$, and also satisfied on B_{ϱ_2}, $|x(0)| \le \varrho$, and $|\phi(0)| \le \delta < \delta^*$, then using the indirect adaptive control law (6), we will ensure that

1) $|\phi(k)|$ will be monotonically nonincreasing, and $|\phi(k) - \phi(k-1)|$ will converge to zero;
2) the tracking error between the plant output and the reference command will converge to a ball of radius ϵ centered at the origin.

Proof: See [12].

III. Auto-Tuning the Adaptation Gain

The gradient update law presented in the previous section relies on the the following idea. Starting with an initial value for the parameter vector, the gradient algorithm changes (updates) this vector by adding to it another vector having a magnitude (that depends on the adaptation gain and the magnitude of the output error) and a direction of descent. We can think of this as searching for the ideal parameter vector. If we keep the adaptation gain fixed (selected *a priori* as presented in [12], [18]), the magnitude of this vector will only depend on the magnitude of the direction of descent and tracking error. Here, however, we argue that the adaptation gain can be selected (adapted) online to minimize $J_u(\eta)$. It is important to mention that our objective here is to search for an "optimal" $\eta(k)$ (that we will call $\eta^{\text{opt}}(k)$). Note that $\eta^{\text{opt}}(k)$ is not necessarily the optimal adaptation gain. The step of finding $\eta^{\text{opt}}(k)$ is crucial to find the new parameter vector $(A^{\text{opt}}(k))$, and hence the new control, $u^{\text{opt}}(k)$. The term optimal is used here only because the adaptation gain (as shown later) will be selected to minimize the the instantaneous control energy $J_u(\eta) = u^2(k)$ that is defined in the second part of the cost function (9). We would like to note that some of the results in this section can be found in [19], [20].

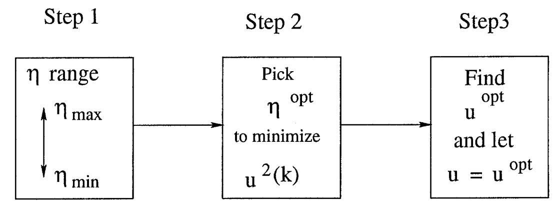

Fig. 1. Steps used for adaptation gain selection.

A. Direct Adaptive Control

The adaptation gain tuning algorithm proceeds according to the following steps (shown in Fig. 1).

1) Find a range on $\eta(k)$ (i.e., $\eta(k) \in [\eta_{\min}(k), \eta_{\max}(k)]$), such that the tracking error is forced to be within an ϵ-neighborhood of zero no matter which $\eta(k)$ in this range is used.
2) Find the new adaptation gain $(\eta^{\text{opt}}(k))$ that minimizes the instantaneous control energy $J_u(k) = u^2(k)$.
3) Using $\eta^{\text{opt}}(k)$, find the new parameter vector $A^{\text{opt}}(k)$ and, hence, the new control $u^{\text{opt}}(k)$.

1) Finding a Feasible Range on $\eta(k)$: The feasible range on $\eta(k)$ is defined in the following theorem.

Theorem 3: Consider the system given in (2). The direct adaptive control law (4) with its parameter update (10) can guarantee stability results stated in Theorem 1 for any adaptation gain $\eta(k) = \alpha(k)\bar{\eta}(k)$ that satisfies $\alpha_1\bar{\eta}(k) \le \eta(k) \le \alpha_2\bar{\eta}(k)$ and $\bar{\eta}(k) = 2[1+\gamma|\zeta(k-d)|^2]/\theta_1|\zeta(k-d)|^2$ if the parameters α_1 and α_2 are selected such that

1) $0 < \alpha_1 \le 1 - (1/2)\rho\theta_1$ and $\rho < 2/\theta_1$, where ρ is a positive design parameter (that is related to the rate of decrease of the Lyapunov function);
2) $\theta_1(2 - \rho\theta_0)/2\theta_0 \le \alpha_2 < 1$.

Proof: Recall that the discrete-time parameter update law is given by (10). Based on our definition of the parameter error ($\phi(k) = A_u(k) - A_u^*$), $\phi(k)$ can be expressed as

$$\phi(k) = \phi(k-1) + \frac{\kappa\eta(k)\zeta(k-d)}{1+\gamma\left|\zeta(k-d)\right|^2}e_\epsilon(k). \quad (14)$$

Consider the Lyapunov-like function $V(k) = \phi^\top(k)\phi(k)$. We will consider the case where $e(k)$ is within the dead zone separate from the case where $e(k)$ is outside the dead zone. First, consider the case where $e(k)$ is inside the dead zone. Then, $e_\epsilon(k) = 0$ so $\phi(k) = \phi(k-1)$ according to (14) and $V(k) - V(k-1) = 0$. With the error outside the dead zone, and for some $\eta(k)$ and $0 < \pi(k) < 1$, we have

$$\begin{aligned} V(k)-V(k-1) &= \phi^\top(k)\phi(k)-\phi^\top(k-1)\phi(k-1) \\ &= \left(\phi(k-1) + \frac{\kappa\eta(k)\zeta(k-d)}{1+\gamma\left|\zeta(k-d)\right|^2}e_\epsilon(k)\right)^\top \\ &\quad \times\left(\phi(k-1)+\frac{\kappa\eta(k)\zeta(k-d)}{1+\gamma\left|\zeta(k-d)\right|^2}e_\epsilon(k)\right) \\ &\quad -\phi^\top(k-1)\phi(k-1) \end{aligned}$$

so that, with (13)

$$V(k)-V(k-1) = \eta(k)\left[\frac{-2}{\pi(k)\theta(k-d)} + \frac{\eta(k)\left|\zeta(k-d)\right|^2}{1+\gamma\left|\zeta(k-d)\right|^2}\right] \times\frac{e_\epsilon^2(k)}{1+\gamma\left|\zeta(k-d)\right|^2}. \tag{15}$$

Since $0 < \pi(k) < 1$ and $0 < \theta_0 \le \theta(k) \le \theta_1$, we get

$$V(k) - V(k-1) \le \eta(k)\left[\frac{-2}{\theta_1} + \frac{\eta(k)\left|\zeta(k-d)\right|^2}{1+\gamma\left|\zeta(k-d)\right|^2}\right] \times\frac{e_\epsilon^2(k)}{1+\gamma\left|\zeta(k-d)\right|^2}. \tag{16}$$

For $(V(k) - V(k-1)) \le 0$, we need

$$0 < \frac{2}{\theta_1} - \frac{\eta(k)\left|\zeta(k-d)\right|^2}{1+\gamma\left|\zeta(k-d)\right|^2}. \tag{17}$$

One way to ensure this is to use $\eta(k)$, where

$$0 < \eta(k) < \frac{2\left[1+\gamma\left|\zeta(k-d)\right|^2\right]}{\theta_1\left|\zeta(k-d)\right|^2} = \bar{\eta}(k) \tag{18}$$

where $\bar{\eta}(k)$ is defined as an upper bound on $\eta(k)$. This implies that stability can be ensured using any value of the adaptation gain

$$\eta(k) = \alpha(k)\bar{\eta}(k) \tag{19}$$

where $0 < \alpha(k) < 1$. Based on this definition of the adaptation gain, the upper and lower bounds on the adaptation gain can be defined as

$$\eta_{\min}(k) = \alpha_1\bar{\eta}(k) \le \eta(k) \le \alpha_2\bar{\eta}(k) = \eta_{\max}(k) \tag{20}$$

where $0 < \alpha_1 \le \alpha(k) \le \alpha_2 < 1$ for fixed constants α_1 and α_2. Later, we will define α_1 and α_2 in terms of known plant information. From the above analysis, stability [i.e., $(V(k) - V(k-1)) < 0$] can be guaranteed for any choice of $\alpha(k)$ such that inequality (20) holds. Next, we study how to choose α_1 and α_2 such that $\alpha(k)$ satisfies the above constraints. Note that if we can find such an $\alpha(k)$, we will be able to derive a range of possible values of $\eta(k)$ that ensures stability. Since $\pi(k) < 1$, (15) becomes

$$V(k) - V(k-1) < \eta(k)\left[\frac{-2}{\theta(k-d)} + \frac{\eta(k)\left|\zeta(k-d)\right|^2}{1+\gamma\left|\zeta(k-d)\right|^2}\right] \times\frac{e_\epsilon^2(k)}{1+\gamma\left|\zeta(k-d)\right|^2}. \tag{21}$$

Suppose we want the decrease of $(V(k) - V(k-1))$ to be influenced by a parameter $\rho > 0$ (ρ is a design parameter we can pick but below we will derive some constraints on its choice), then $\eta(k)$ that meets this requirement can be expressed as

$$0 < \rho = \frac{2}{\theta(k-d)} - \frac{\eta(k)\left|\zeta(k-d)\right|^2}{1+\gamma\left|\zeta(k-d)\right|^2}. \tag{22}$$

Then, $\eta(k)$ can be written as

$$\eta(k) = \frac{\left[2-\rho\theta(k-d)\right]\left[1+\gamma\left|\zeta(k-d)\right|^2\right]}{\theta(k-d)\left|\zeta(k-d)\right|^2}. \tag{23}$$

Since $\eta(k) = \alpha(k)\bar{\eta}(k)$, then

$$\alpha(k) = \frac{\theta_1\left[2-\rho\theta(k-d)\right]}{2\theta(k-d)}. \tag{24}$$

Notice that since $0 < \theta_0 \le \theta(k) \le \theta_1$, the smallest $\alpha(k)$ will be is $\alpha_{\min}$ where

$$\alpha_{\min} = \frac{\theta_1(2-\rho\theta_1)}{2\theta_1} = 1 - \frac{1}{2}\rho\theta_1. \tag{25}$$

We will pick α_1 such that

$$0 < \alpha_1 \le 1 - \frac{1}{2}\rho\theta_1 \tag{26}$$

and

$$\rho < \frac{2}{\theta_1} \tag{27}$$

since we want to guarantee that $\alpha_{\min} > 0$. Note also that the largest $\alpha(k)$ will be is $\alpha_{\max}$ where

$$\alpha_{\max} = \frac{\theta_1(2-\rho\theta_0)}{2\theta_0} \tag{28}$$

and we pick α_2 such that $\alpha_{\max} \le \alpha_2 < 1$ so

$$\frac{\theta_1(2-\rho\theta_0)}{2\theta_0} \le \alpha_2 < 1. \tag{29}$$

Note that for the previous oices, we know that $0 < \alpha_1 < \alpha_2 < 1$ because

$$1 > \alpha_2 \ge \frac{\theta_1(2-\rho\theta_0)}{2\theta_0} > \frac{2-\rho\theta_0}{2} > \frac{2-\rho\theta_1}{2} \ge \alpha_1 > 0. \tag{30}$$

With this we know that there will be a range of possible $\alpha(k)$ values. At this stage of our analysis, it is important to study the feasibility of our choices of α_1 and α_2 (in terms of plant dynamics). One way to make such a study is to see how big

$\alpha_{2\min} - \alpha_{1\max}$ is, where $\alpha_{2\min}$ and $\alpha_{1\max}$ are the minimum and maximum values that α_2 and α_1 can have

$$\alpha_{2\min} - \alpha_{1\max} = \frac{\theta_1(2-\rho\theta_0)}{2\theta_0} - \frac{\theta_0(2-\rho\theta_1)}{2\theta_0} = \frac{\theta_1-\theta_0}{\theta_0}. \quad (31)$$

Therefore, for this approximation to apply, we need

$$0 < \frac{\theta_1-\theta_0}{\theta_0} < 1 \quad (32)$$

which means that $0 < (\theta_1 - \theta_0) < \theta_0$ since $0 < \alpha_2 - \alpha_1 < 1$. Hence, this is very suitable for applications where we want to find as small of θ_1 as possible and as big of θ_0 as possible. ∎

Note that here, we picked ρ and found the resulting values of α_1 and α_2. We can also start by picking α_1 and α_2 such that

$$0 < \alpha_1 < \alpha_2 < 1 \quad (33)$$

with

$$\rho < \frac{2}{\theta_1} \quad (34)$$

then pick ρ such that

$$2\alpha_1 \le (2-\rho\theta_1) \quad (35)$$

which can be written as

$$\rho \le \frac{2-2\alpha_1}{\theta_1} \quad (36)$$

and also so that

$$2\alpha_2\theta_0 \ge \theta_1(2-\rho\theta_0) \quad (37)$$

or, equivalently

$$\rho \ge \frac{2\theta_1 - 2\theta_0\alpha_2}{\theta_0\theta_1}. \quad (38)$$

Therefore, ρ should be chosen such that

$$\frac{2\theta_1 - 2\theta_0\alpha_2}{\theta_0\theta_1} \le \rho \le \frac{2-2\alpha_1}{\theta_1} \quad (39)$$

and also

$$0 < \rho < \frac{2}{\theta_1}. \quad (40)$$

Now, it is important to study the feasibility of our choice of ρ. We can do that by finding the constraints that guarantee two things. First, the upper bound of ρ is always greater than the lower bound. Second, $\rho > 0$. To check the first condition, we want to make sure that (39) holds. That is

$$\begin{aligned} 2\theta_1 - 2\theta_0\alpha_2 &< 2\theta_0 - 2\theta_0\alpha_1 \\ 2(\theta_1-\theta_0) &< 2\theta_0(\alpha_2-\alpha_1) \\ \frac{\theta_1-\theta_0}{\theta_0} &< (\alpha_2-\alpha_1). \end{aligned}$$

So, it is clear that for the upper bound of ρ to be always greater than the lower bound, we need

$$\frac{\theta_1}{\theta_0} - 1 < (\alpha_2 - \alpha_1). \quad (41)$$

Also, to make sure that $\rho > 0$, it can be shown that we need the lower bound of (39) be greater than zero or, equivalently, the following inequality be satisfied:

$$\alpha_2 < \frac{\theta_1}{\theta_0}. \quad (42)$$

Next, we will show that using the gradient update law (whose adaptation gain is adapted to satisfy the above requirements) along with the continuous dead zone, the output error is forced to stay within an ϵ-neighborhood of zero.

2) Finding the New Adaptation Gain $\eta^{\text{opt}}(k)$ via Minimizing the Instantaneous Control Energy: Here, the new adaptation gain is obtained by minimizing the second part of the cost function (9) which can be expressed as

$$\min \quad J_u(\eta) = u^2(k) \quad (43)$$

such that $\eta_{\min}(k) \le \eta(k) \le \eta_{\max}(k)$. The control defined in (4) (assuming $u_k(k) = 0$, we have no prior information about the ideal control), can be written as

$$u(k) = A^\top(k-1)\zeta(k) + \frac{\kappa\eta\zeta(k-d)^\top\zeta(k)}{1+\gamma\left|\zeta(k-d)\right|^2}e_\epsilon(k). \quad (44)$$

Using (44), it can be shown that $u^2(k)$ can be written as

$$u^2(k) = T_1(k)\eta^2(k) + T_2(k)\eta(k) + T_3(k) \quad (45)$$

where

$$T_1(k) = \frac{2e_\epsilon^2(k)\left[\zeta(k-d)^\top\zeta(k)\right]^2}{\left[1+\gamma\left|\zeta(k-d)\right|^2\right]^2}$$

$$T_2(k) = \frac{2A^\top(k-1)\zeta(k)e_\epsilon\zeta(k-d)^\top\zeta(k)}{1+\gamma\left|\zeta(k-d)\right|^2}$$

and

$$T_3(k) = \left[A^\top(k-1)\zeta(k)\right]^2.$$

Since $T_3(k)$ in independent of $\eta(k)$ it can be omitted. Since $u^2(k)$ expressed in (45) is in quadratic form, the cost function (43) can be minimized as a quadratic programming problem with linear inequality constraint ($\eta_{\min}(k) \le \eta(k) \le \eta_{\max}(k)$). Since $T_1(\mathrm{k})$ is positive definite, this problem is known to have a unique global minimum, $\eta^{\text{opt}}(k)$, which is used to find the new parameter vector and, hence, the new control. Now, this adaptation gain can be used in the update routine of the controller's parameter vector as shown next.

The new adaptation gain $\eta^{\text{opt}}(k)$ can be used to find the new parameter vector $A^{\text{opt}}(k)$ as follows:

$$A^{\text{opt}}(k) = A^{\text{opt}}(k-1) + \frac{\eta^{\text{opt}}(k)\zeta(k-d)}{1+\gamma\left|\zeta(k-d)\right|^2}e_\epsilon(k). \quad (46)$$

This new parameter vector of the controller is used to find the new control as

$$u^{\text{opt}}(k) = A^{\text{opt}}(k)^T\zeta(k) \quad (47)$$

which is the control to be input to the system.

It is important at this point to show that $u^{\text{opt}}(k)$ (that is found using $\eta^{\text{opt}}(k) \in [\eta_{\min}, \eta_{\max}]$) lies inside the feasible control range $[u_{\min}(k), u_{\max}(k)]$. This is shown in the next theorem.

Theorem 4: Given that the new adaptation gain ($\eta^{\mathrm{opt}}(k)$) is defined such that $0 < \eta_{\min}(k) \leq \eta^{\mathrm{opt}}(k) \leq \eta_{\max}(k)$, the direct adaptive control law ($u^{\mathrm{opt}}(k)$ that is obtained using $\eta^{\mathrm{opt}}(k)$) lies inside the feasible control range ($[u_{\min}(k), u_{\max}(k)]$).

Proof: Since the control is defined as $u(k) = A^{\top}(k)\zeta(k)$, it can be shown that the control can be written as

$$u(k) = N_1(k) + \eta(k)N_2(k) \tag{48}$$

where $N_1(k) = A^{\top}(k-1)\zeta(k)$ and $N_2(k) = \zeta(k-d)^{\top}\zeta(k)e_{\epsilon}(k)/(1+\gamma|\zeta(k-d)|^2)$. We know from the previous sections that $\eta_{\min}(k)$ and $\eta_{\max}(k)$ do not necessarily produce $A_{\min}(k)$ and $A_{\max}(k)$, respectively. Hence, $u_{\max}(k)$ and $u_{\min}(k)$ are defined as

$$u_{\max}(k) = \begin{cases} N_1(k) + \eta_{\max}(k)N_2(k), & \text{if } \eta^{\max}(k) \text{ produces } A_{\max}(k) \\ N_1(k) + \eta_{\min}(k)N_2(k), & \text{if } \eta^{\min}(k) \text{ produces } A_{\max}(k) \end{cases} \tag{49}$$

and

$$u_{\min}(k) = \begin{cases} N_1(k) + \eta_{\min}(k)N_2(k), & \text{if } \eta^{\min}(k) \text{ produces } A_{\min}(k) \\ N_1(k) + \eta_{\max}(k)N_2(k), & \text{if } \eta^{\max}(k) \text{ produces } A_{\min}(k) \end{cases}. \tag{50}$$

It is clear at this point that we have two cases. In the first case, $A_{\max}(k)$ and $A_{\min}(k)$ are found using $\eta^{\max}(k)$ and $\eta^{\min}(k)$, respectively. In the second case, however, $A_{\max}(k)$ and $A_{\min}(k)$ are found using $\eta^{\min}(k)$ and $\eta^{\max}(k)$, respectively. It can be easily shown that $N_2(k)$ is positive in the first case and negative in the second. To see this, note that in the first case we know that

$$u_{\min} \leq u_{\max} \tag{51}$$

or, equivalently

$$N_1(k) + \eta_{\min}(k)N_2(k) \leq N_1(k) + \eta_{\max}(k)N_2(k). \tag{52}$$

Subtracting $N_1(k)$ from both sides, we get

$$\eta_{\min}(k)N_2(k) \leq \eta_{\max}(k)N_2(k). \tag{53}$$

Hence, we have

$$0 \leq (\eta_{\max}(k) - \eta_{\min}(k))\, N_2(k) \tag{54}$$

which means that $N_2(k)$ is positive, knowing (by definition) that $\eta_{\max}(k) \geq \eta_{\min}(k)$. In the second case, a similar way can be used to show that $N_2(k)$ is negative.

Let us consider each case separately.

Case 1) ($N_2(k) > 0$)

By definition, we know that

$$u_{\min}(k) \leq u_{\max}(k) \tag{55}$$

or equivalently

$$N_1(k) + \eta_{\min}(k)N_2(k) \leq N_1(k) + \eta_{\max}(k)N_2(k). \tag{56}$$

We are given that $0 < \eta_{\min}(k) \leq \eta^{\mathrm{opt}}(k) \leq \eta_{\max}(k)$. Since $N_2(k) > 0$, we have

$$\eta_{\min}(k)N_2(k) \leq \eta^{\mathrm{opt}}(k)N_2(k) \leq \eta_{\max}(k)N_2(k). \tag{57}$$

Adding $N_1(k)$ to the previous inequality, we get

$$\begin{aligned} N_1(k) + \eta_{\min}(k)N_2(k) &\leq N_1(k) + \eta^{\mathrm{opt}}(k)N_2(k) \\ &\leq N_1(k) + \eta_{\max}(k)N_2(k) \end{aligned}$$

or

$$u_{\min}(k) \leq u^{\mathrm{opt}}(k) \leq u_{\max}(k). \tag{58}$$

Now, consider the second case.

Case 2) ($N_2(k) < 0$)

By definition, we know that

$$u_{\min}(k) \leq u_{\max}(k) \tag{59}$$

or, equivalently

$$N_1(k) + \eta_{\max}(k)N_2(k) \leq N_1(k) + \eta_{\min}(k)N_2(k). \tag{60}$$

We are given that $0 < \eta_{\min}(k) \leq \eta^{\mathrm{opt}}(k) \leq \eta_{\max}(k)$. Since $N_2(k) < 0$, we have

$$\eta_{\min}(k)N_2(k) \geq \eta^{\mathrm{opt}}(k)N_2(k) \geq \eta_{\max}(k)N_2(k). \tag{61}$$

Adding $N_1(k)$ to the previous inequality, we get

$$\begin{aligned} N_1(k) + \eta_{\min}(k)N_2(k) &\geq N_1(k) + \eta^{\mathrm{opt}}(k)N_2(k) \\ &\geq N_1(k) + \eta_{\max}(k)N_2(k) \end{aligned}$$

or

$$u_{\max}(k) \geq u^{\mathrm{opt}}(k) \geq u_{\min}(k) \tag{62}$$

which completes the proof.

B. Indirect Adaptive Control

The general steps in this adaptation gain tuning algorithm are the same as the one outlined in the direct case; however, there are some key differences in the stability conditions and the deter\mi-nation of the control law. Let us start by discussing the steps of the algorithm which are summarized in Fig. 1.

1) Finding a Feasible Range on $\eta(k)$: Before stating the theorem that defines the feasible range on the adaptation gain, we present the following assumption.

Assumption 1: It is assumed that a lower bound on $|\zeta(k)|^2$ is known, that is $|\zeta(k)|^2 \geq \bar{\zeta}$.

Theorem 5: Consider the system given in (2). The indirect adaptive control law (6) with its parameter update law (10) can guarantee stability results stated in Theorem (2) for any adaptation gain $\eta(k) = \alpha(k)\bar{\eta}(k)$ that satisfies Assumption 1, $\alpha_1\bar{\eta}(k) \leq \eta(k) \leq \alpha_2\bar{\eta}(k)$, and

$$\bar{\eta}(k) = \begin{cases} 2\left[\frac{1}{\bar{\zeta}} + \gamma\right], & \text{if } \bar{\eta}(k) \geq 2\left[\frac{1}{\bar{\zeta}} + \gamma\right] \\ 2\left[\frac{1}{|\zeta(k-d)|^2} + \gamma\right], & \text{if } \bar{\eta}(k) < 2\left[\frac{1}{\bar{\zeta}} + \gamma\right] \end{cases} \tag{63}$$

if the parameters α_1 and α_2 are selected such that $0 < \alpha_1 \leq \alpha(k) \leq \alpha_2 < 1$.

Proof: Based on our definition of the parameter error ($\phi(k) = A_u(k) - A_u^*$), $\phi(k)$ can be expressed as

$$\phi(k) = \phi(k-1) + \frac{\kappa\eta(k)\zeta(k-d)}{1+\gamma\left|\zeta(k-d)\right|^2}e_{\epsilon}(k). \tag{64}$$

Consider the Lyapunov-like function $V(k) = \phi^\top(k)\phi(k)$. The case where $e(k)$ is within the dead zone is the same as before. With error outside the dead zone, and for some $\eta(k)$ and $0 < \pi(k) < 1$, we have

$$\begin{aligned}V(k)-V(k-1) &= \phi^\top(k)\phi(k)-\phi^\top(k-1)\phi(k-1)\\ &= \left(\phi(k-1)+\frac{\kappa\eta(k)\zeta(k-d)}{1+\gamma\left|\zeta(k-d)\right|^2}e_\epsilon(k)\right)^\top\\ &\quad\times\left(\phi(k-1)+\frac{\kappa\eta(k)\zeta(k-d)}{1+\gamma\left|\zeta(k-d)\right|^2}e_\epsilon(k)\right)\\ &\quad-\phi^\top(k-1)\phi(k-1)\end{aligned}$$

so that, with (13), we have

$$V(k)-V(k-1) = \eta(k)\left[\frac{-2}{\pi(k)}+\frac{\kappa^2\eta(k)\left|\zeta(k-d)\right|^2}{1+\gamma\left|\zeta(k-d)\right|^2}\right]\times\frac{e_\epsilon^2(k)}{1+\gamma\left|\zeta(k-d)\right|^2}. \quad (65)$$

Since $0 < \pi(k) < 1$ and $\kappa^2 = 1$ (since $\kappa = -1$ in this case), we get

$$V(k)-V(k-1) \leq \eta(k)\left[-2+\frac{\eta(k)\left|\zeta(k-d)\right|^2}{1+\gamma\left|\zeta(k-d)\right|^2}\right]\times\frac{e_\epsilon^2(k)}{1+\gamma\left|\zeta(k-d)\right|^2}. \quad (66)$$

Note that (66) is similar to (16), except that $\theta_1 = 1$ (since in this case $\theta(k) = 1$). It can be shown that $\eta(k)$ can be defined the same way as in (20), where $\bar{\eta}(k)$ is defined as

$$\bar{\eta}(k) = \frac{2\left[1+\gamma\left|\zeta(k-d)\right|^2\right]}{\left|\zeta(k-d)\right|^2} \quad (67)$$

From the previous analysis, stability can be guaranteed for any choice of $\alpha(k)$ such that $\alpha_1\bar{\eta}(k) \leq \eta(k) \leq \alpha_2\bar{\eta}(k)$ holds. Also, it can be shown that the parameter ρ can be expressed as

$$0 < \rho = 2 - \frac{\eta(k)\left|\zeta(k-d)\right|^2}{1+\gamma\left|\zeta(k-d)\right|^2} \quad (68)$$

Then, $\eta(k)$ can be written as

$$\eta(k) = \frac{[2-\rho]\left[1+\gamma\left|\zeta(k-d)\right|^2\right]}{\left|\zeta(k-d)\right|^2}. \quad (69)$$

Since $\eta(k) = \alpha(k)\bar{\eta}(k)$, then

$$\alpha(k) = \frac{2-\rho}{2}. \quad (70)$$

To ensure that $0 < \alpha(k) < 1$, the rate of decrease has to be bounded within $0 < \rho < 2$. It is clear from this discussion that unlike the direct case, the choice of the bounds of α in the indirect case is independent of the plant dynamics. However, no matter what choice we make in selecting the bounds of α (as long as $0 < \alpha_1 < \alpha_2 < 1$), the a rate of decrease of $V(k) - V(k-1)$ is confined in the interval (0,2) (i.e., $0 < \rho < 2$). These conclusions represent the major differences between the two cases.

Notice that since $|\zeta(k)|^2$ can be very small (as in TSFSs), it is important to assume that we know a lower bound on $|\zeta(k)|^2$ (i.e., $|\zeta(k)|^2 \geq \bar{\zeta}$). Such a lower bound on $|\zeta(k)|^2$ provides an upper bound on the adaptation gain. It is known that the maximum possible adaptation gain is given by

$$\bar{\eta}(k) = 2\left[\frac{1}{\left|\zeta(k-d)\right|^2}+\gamma\right]. \quad (71)$$

Using (71) and the assumption that $|\zeta(k)|^2 \geq \bar{\zeta}$, then the upper bound on $\bar{\eta}(k)$ becomes

$$\bar{\eta}(k) \leq 2\left[\frac{1}{\bar{\zeta}}+\gamma\right]. \quad (72)$$

This can be guaranteed by letting

$$\bar{\eta}(k) = \begin{cases} 2\left[\frac{1}{\bar{\zeta}}+\gamma\right], & \text{if } \bar{\eta}(k) \geq 2\left[\frac{1}{\bar{\zeta}}+\gamma\right] \\ 2\left[\frac{1}{|\zeta(k-d)|^2}+\gamma\right], & \text{if } \bar{\eta}(k) < 2\left[\frac{1}{\bar{\zeta}}+\gamma\right] \end{cases}. \quad (73)$$

This upper bound on the maximum adaptation gain can be used to ensure boundedness of the new adaptation gain, $\eta^{\text{opt}}(k)$. ■

Next, we will show how to select the adaptation gain to minimize the instantaneous control energy.

2) Finding the New Adaptation Gain $(\eta^{\text{opt}}(k))$: Since $\kappa = -1$ in the indirect case, the instantaneous control becomes

$$u(k) = A^\top(k-1)\zeta(k) - \frac{\eta\zeta(k-d)^\top\zeta(k)}{1+\gamma\left|\zeta(k-d)\right|^2}e_\epsilon(k) \quad (74)$$

and, hence, $u^2(k)$ becomes

$$u^2(k) = T_1(k)\eta^2(k) + T_2(k)\eta(k) + T_3(k) \quad (75)$$

where

$$T_1(k) = \frac{2e_\epsilon^2(k)\left[\zeta(k-d)^\top\zeta(k)\right]^2}{\left[1+\gamma\left|\zeta(k-d)\right|^2\right]^2}$$

$$T_2(k) = \frac{-2A^\top(k-1)\zeta(k)e_\epsilon\zeta(k-d)^\top\zeta(k)}{1+\gamma\left|\zeta(k-d)\right|^2}$$

and

$$T_3(k) = \left[A^\top(k-1)\zeta(k)\right]^2.$$

Since $T_3(k)$ in independent of $\eta(k)$ it can be omitted; hence, the cost function can be solved as quadratic programming problem with linear inequality constraints which is known to have a unique optimal solution since $T_1(k)$ is positive definite. Once the new adaptation gain is found, it can be used to find the new parameter vector $A^{\text{opt}}(k)$ as follows:

$$A^{\text{opt}}(k) = A^{\text{opt}}(k-1) - \frac{\eta^{\text{opt}}(k)\zeta(k-d)}{1+\gamma\left|\zeta(k-d)\right|^2}e_\epsilon(k). \quad (76)$$

The new parameter vector can be used to find approximation of the plant dynamics as follows:

$$\alpha^{\text{opt}}(k) = A_\alpha^{\text{opt}\,\top}(k)\zeta_\alpha(k) \quad (77)$$

and

$$\beta^{\text{opt}}(k) = A_\beta^{\text{opt}\,\top}(k)\zeta_\beta(k). \quad (78)$$

Now, the new control to be input to the system can be easily computed by

$$u^{\rm opt}(k) = \frac{-\alpha^{\rm opt}(k) + r(k)}{\beta^{\rm opt}(k)}. \tag{79}$$

It is important at this point to show that $u^{\rm opt}(k)$ (that is found using $\eta^{\rm opt}(k) \in [\eta_{\min}, \eta_{\max}]$) lies inside the feasible control range $[u_{\min}(k), u_{\max}(k)]$.

Theorem 6: Given that the new adaptation gain $(\eta^{\rm opt}(k))$ is defined such that $0 < \eta_{\min}(k) \leq \eta^{\rm opt}(k) \leq \eta_{\max}(k)$, the indirect adaptive control law ($u^{\rm opt}(k)$ that is obtained using $\eta^{\rm opt}(k)$) lies inside the feasible control range $([u_{\min}(k), u_{\max}(k)])$.

Proof: The proof is the same as the one used to prove Theorem 4. ■

IV. Auto-Tuning the Direction of Descent

As discussed earlier, the gradient update routine is based on the idea that starting with an initial value for the parameter vector, the gradient algorithm changes (updates) this vector by adding to it another vector having a magnitude and a direction of descent. We can think of this as searching for the ideal parameter vector. To improve (or loosely speaking, attempt to "optimize") the performance of the searching mechanism in the gradient-based update law, we will attempt to modify the direction of descent so that a certain cost criterion of interest is optimized. Note that this approach is applicable to both direct and indirect adaptive control schemes. Next, we will present our assumption.

A. Assumptions

Assumption 2: The new direction of descent vector $\zeta^*(k)$ can be expressed as

$$\zeta^*(k) = \zeta(k) + \zeta_e(k) \tag{80}$$

where $\zeta_e(k)$ is defined as an incremental vector that when it is added to the no\minal directional vector $\zeta(k)$, $\zeta^*(k)$ is obtained. We also assume that

$$-\bar{\sigma}\,{\rm abs}\left[\zeta(k)\right] \leq \zeta_e(k) \leq \bar{\sigma}\,{\rm abs}\left[\zeta(k)\right] \tag{81}$$

where $\zeta(k)$ and ${\rm abs}[\zeta(k)] \in \mathcal{R}^p$ are defined as

$$\zeta(k) = \begin{bmatrix} \zeta_1(k) \\ \zeta_2(k) \\ \vdots \\ \zeta_p(k) \end{bmatrix} \tag{82}$$

and

$${\rm abs}\left[\zeta(k)\right] = \begin{bmatrix} {\rm abs}\left[\zeta_1(k)\right] \\ {\rm abs}\left[\zeta_2(k)\right] \\ \vdots \\ {\rm abs}\left[\zeta_p(k)\right] \end{bmatrix} \tag{83}$$

respectively.

Assumption 3: We assume that $|\zeta(k)|$ has a known upper bound $\zeta_{\max}$ (i.e., $|\zeta(k)| \leq \zeta_{\max}, \forall k = 0, 1, 2, \ldots$).

Now, we will present our proposed algorithm for auto-tuning the direction of descent.

B. Algorithm Description

Using the previous assumptions, our goal here is to auto-tune the direction of descent vector that is characterized by $\zeta(k)$ so that an improved directional vector $\zeta^*(k)$ is obtained. Recall the gradient update law defined in (10). Substituting (80) into the update law defined in (10), we get

$$A(k) = A(k-1) + \frac{\kappa\eta e_\epsilon(k)\left[\zeta(k-d) + \zeta_e(k-d)\right]}{1 + \gamma\left|\zeta(k-d) + \zeta_e(k-d)\right|^2}. \tag{84}$$

To simplify notation, let $\varphi(k) = \kappa\eta e_\epsilon(k)/(1 + \gamma|\zeta(k-d) + \zeta_e(k-d)|^2)$, then (84) becomes

$$A(k) = A(k-1) + \varphi(k)\left[\zeta(k-d) + \zeta_e(k-d)\right]. \tag{85}$$

Based on the definition of the parameter error ($\phi(k) = A(k) - A^*$), $\phi(k)$ can be expressed as

$$\phi(k) = \phi(k-1) + \varphi(k)\left[\zeta(k-d) + \zeta_e(k-d)\right]. \tag{86}$$

One choice of a cost criterion that we wish to optimize is the control energy defined as $J_u = u^2(k)$. Thus, we need a new formulation of $u(k)$ in which the new definition of the directional vector is incorporated. Based on (80), $u(k)$ can be defined as

$$u(k) = A^\top(k)\left[\zeta(k) + \zeta_e(k)\right]. \tag{87}$$

Substituting (85) into (87), $u(k)$ can be written as

$$u(k) = \Psi(k)\zeta_e(k) + \Phi(k) \tag{88}$$

where

$$\Psi(k) = A^\top(k-1) + \varphi(k)\zeta^\top(k-d) + \varphi(k)\zeta_e^\top(k-d)$$

and

$$\Phi(k) = A^\top(k-1)\zeta(k) + \varphi(k)\zeta^\top(k-d)\zeta(k) + \varphi(k)\zeta_e^\top(k-d)\zeta(k).$$

Based on this formulation, it can be shown that the instantaneous control energy function $(J_u = u^2(k))$ can be written as

$$J_u = u^2(k) = \zeta_e^\top(k)H_1(k)\zeta_e(k) + 2H_2(k)\zeta_e(k) + H_3(k) \tag{89}$$

where $H_1(k) = \Psi^\top(k)\Psi(k)$, $H_2(k) = \Phi(k)\Psi(k)$, and $H_3(k) = \Phi^\top(k)\Phi(k)$. Since $H_3(k)$ is not defined in terms of $\zeta_e(k)$, it can be omitted from the function we need to optimize. Thus, to find $\zeta_e(k)$ we need to solve the following cost function:

$$\min\ J_u(\zeta_e) = \min\ \left\{\zeta_e^\top(k)H_1(k)\zeta_e(k) + 2H_2(k)\zeta_e(k)\right\}. \tag{90}$$

It is known that the cost function (90) has a unique solution when it is solved subject to a linear constraint. In this case the cost function $J_u(\zeta_e)$ can be solved subject to the linear constraint (81) defined in Assumption 2. Hence, to solve for $\zeta_e(k)$, the following cost function needs to be minimized:

$$J_u = \min\ \left\{\zeta_e^\top(k)H_1(k)\zeta_e(k) + 2H_2(k)\zeta_e(k)\right\} \quad s.t. -\bar{\sigma}\,{\rm abs}\left[\zeta(k)\right] \leq \zeta_e(k) \leq \bar{\sigma}\,{\rm abs}\left[\zeta(k)\right]. \tag{91}$$

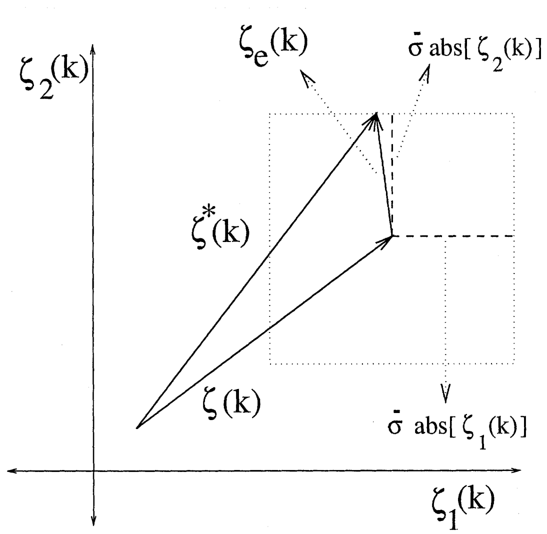

Fig. 2. Two-dimensional geometrical presentation.

At this point, we have derived a direction of descent that minimizes not only the squared output error, but also the control energy. Substituting the direction of descent $\zeta^*(k)$ in (10), a new parameter vector (that uses $\zeta^*(k)$ as its new direction of descent) can be obtained. Finally, this new direction of descent, $\zeta^*(k)$, can be used to deter\mine the new parameter vector [using $\zeta(k) = \zeta^*(k)$ in (10)] and new control [using (4) and (6) in the direct and indirect cases, respectively].

C. Geometrical Interpretation and Relation to Auto-Tuning the Adaptation Gain

In Assumption 2, we assumed that the new direction of descent can be expressed by (80), where $\zeta_e(k)$ is defined as an incremental vector that when it is added the no\minal directional vector $\zeta(k)$, an improved vector $\zeta^*(k)$ is obtained. To describe this assumption geometrically, consider (for simplicity) the two dimensional case (i.e., $\zeta(k) \in \mathcal{R}^2$). This situation can be shown in Fig. 2. We know (from Assumption 2) that

$$\zeta^*(k) = \zeta(k) + \zeta_e(k). \tag{92}$$

It can be verified that $\zeta^*(k)$ can be expressed as

$$\zeta^*(k) = \zeta(k) + \sigma(k)\zeta(k) \tag{93}$$

where

$$\sigma(k) = \begin{bmatrix} \sigma_1(k) & 0 & & 0 \\ 0 & \sigma_2(k) & & \cdot \\ \cdot & 0 & \ddots & \cdot \\ \cdot & \cdot & & 0 \\ 0 & 0 & & \sigma_p(k) \end{bmatrix} \tag{94}$$

and $-\bar{\sigma} \leq \sigma_i(k) \leq \bar{\sigma}, \forall i = 1, 2, \ldots, p$. Equation (93) can be written as

$$\zeta^*(k) = \Lambda(k)\zeta(k) \tag{95}$$

where $\Lambda(k) = I + \sigma(k)$ and I is the identity matrix. Note that the diagonal matrices $\Lambda(k)$, $\sigma(k)$, and $I \in \mathcal{R}^{p \times p}$. It is clear that $\Lambda(k)$, which can be expressed as

$$\begin{aligned} \Lambda(k) &= \begin{bmatrix} 1+\sigma_1(k) & 0 & & 0 \\ 0 & 1+\sigma_2(k) & & \cdot \\ \cdot & 0 & \ddots & \cdot \\ \cdot & \cdot & & 0 \\ 0 & 0 & & 1+\sigma_p(k) \end{bmatrix} \\ &= \begin{bmatrix} \Lambda_1(k) & 0 & & 0 \\ 0 & \Lambda_2(k) & & \cdot \\ \cdot & 0 & \ddots & \cdot \\ \cdot & \cdot & & 0 \\ 0 & 0 & & \Lambda_p(k) \end{bmatrix} \end{aligned} \tag{96}$$

is a diagonal gain matrix that when it is multiplied by the no\minal direction of descent, a new vector $\zeta^*(k)$ is obtained. From the definition of $\sigma(k)$, we know that

$$\begin{aligned} -\bar{\sigma} &\leq \sigma_i(k) \leq \bar{\sigma} \\ 1 - \bar{\sigma} &\leq \sigma_i(k) + 1 \leq \bar{\sigma} + 1 \\ 1 - \bar{\sigma} &\leq \Lambda_i(k) \leq 1 + \bar{\sigma} \qquad \forall i = 1, 2, \ldots, p. \end{aligned} \tag{97}$$

Note that for the special case when

$$\Lambda_1(k) = \Lambda_2(k) = \cdots = \Lambda_p(k). \tag{98}$$

$\zeta^*(k)$ will be a scaled version of $\zeta(k)$, that is, we only change the magnitude of the direction of descent vector. It is clear that this special case is equivalent to auto-tuning the adaptation gain (presented in [19] and [20]). However, when the condition (98) is not satisfied, some type of change will occur to the direction of the direction of descent vector.

As a result of this algorithm, the overall control energy is expected to decrease as $\bar{\sigma}$ increases since $\zeta^*(k)$ will be affected by $\zeta_e(k)$, which is found by minimizing the instantaneous control energy. It is interesting to note that the line of argument used here to select the magnitude of $\bar{\sigma}$ is similar to one used in selecting the gains of the quadratic cost function $(x^\top(k)Qx(k) + u^\top(k)Ru(k))$ used in linear quadratic regulation problems, where the gains Q and R are selected based on the desired closed-loop performance relative to the affordable control energy.

Under the special case when $\bar{\sigma} = 1$, $\Lambda(k)$ (and, hence, the control) equals zero if

$$\Lambda_1(k) = \Lambda_2(k) = \cdots = \Lambda_p(k) = 0 \tag{99}$$

which is equivalent to

$$\sigma_1(k) = \sigma_2(k) = \cdots = \sigma_p(k) = -1. \tag{100}$$

Hence, to avoid this special which may result in a zero control, one may want to exclude the case when $\bar{\sigma} = 1$.

D. Stability Analysis

For both direct and indirect cases, we present the following result.

Theorem 7: Suppose $|r(k)| \leq d_1$ for all $k \geq 0$. Given any constant $\varrho > 0$ and any small constant $\epsilon > 0$, there exist positive constants $\varrho_1 = \varrho_1(\varrho, d_1)$, $\varrho_2 = \varrho_2(\varrho, d_1)$, $\varepsilon^* = \varepsilon^*(\varrho, \epsilon, d_1)$, and $\delta^* = \delta^*(\varrho, \epsilon, d_1)$ such that if the appropriate assumptions stated in [12] are satisfied on $\mathcal{S}_x \supset B_{\varrho_1}$ with $\varepsilon < \varepsilon^*$ and on B_{ϱ_2}, Assumptions 2 and 3 are satisfied for all $k \geq 0$, $|x(0)| \leq \varrho$, $|\phi(0)| \leq \delta < \delta^*$, and the parameters η and γ are selected to satisfy

$$0 < (\eta\theta_1 - 2\gamma) < \frac{2}{(\bar{\sigma}+2)^2\zeta_{\max}^2} \tag{101}$$

then using either the direct (4) or indirect (6) adaptive control law with the direction of descent selected by solving (91), we will ensure that

1) $|\phi(k)|$ will be monotonically nonincreasing, and $|\phi(k) - \phi(k-1)|$ will converge to zero;
2) the tracking error between the plant output and the reference command will converge to a ball of radius ϵ centered at the origin.

Proof: The proof of this theorem is similar to the proof of Theorem 1 in the direct case (or Theorem 2 in the indirect case), except for the part in step 2) where we need to show that the Lyapunov-like function $V(k) = \phi^\top(k)\phi(k)$ is monotonically nonincreasing. We will consider the case where $e(k)$ is within the dead zone separate from the case where $e(k)$ is outside the dead zone. First, consider the case where $e(k)$ is inside the dead zone. In this case, $e_\epsilon(k) = 0$ so $\phi(k) = \phi(k-1)$ and $V(k) - V(k-1) = 0$. With the error outside the dead zone, we have

$$\begin{aligned} V(k) &- V(k-1) \\ &= \phi^\top(k)\phi(k) - \phi^\top(k-1)\phi(k-1) \\ &= \left(\phi(k-1) + \frac{\kappa\eta\left[\zeta(k-d)+\zeta_e(k-d)\right]}{1+\gamma\left|\zeta(k-d)+\zeta_e(k-d)\right|^2}e_\epsilon(k)\right)^\top \\ &\quad\times\left(\phi(k-1) + \frac{\kappa\eta\left[\zeta(k-d)+\zeta_e(k-d)\right]}{1+\gamma\left|\zeta(k-d)+\zeta_e(k-d)\right|^2}e_\epsilon(k)\right) \\ &\quad - \phi^\top(k-1)\phi(k-1). \end{aligned} \tag{102}$$

Based on (13), it can be shown that

$$e_\epsilon(k) = -\kappa\pi(k)\theta(k-d)\phi(k-d)^\top\left[\zeta(k-d)+\zeta_e(k-d)\right]. \tag{103}$$

Using (103) and the fact that $\kappa^2 = 1$, (102) can be expressed as

$$\begin{aligned} V(k) &- V(k-1) \\ &= \eta\left[\frac{-2}{\pi(k)\theta(k-d)} + \frac{\eta\left|\zeta(k-d)+\zeta_e(k-d)\right|^2}{1+\gamma\left|\zeta(k-d)+\zeta_e(k-d)\right|^2}\right] \\ &\quad\times\frac{e_\epsilon^2(k)}{1+\gamma\left|\zeta(k-d)+\zeta_e(k-d)\right|^2}. \end{aligned} \tag{104}$$

Since $0 < \pi(k) < 1$ and $0 < \theta_0 \leq \theta(k-d) \leq \theta_1$, we get

$$V(k) - V(k-1) \leq \eta\left[\frac{-2}{\theta_1} + \frac{\eta\left|\zeta(k-d)+\zeta_e(k-d)\right|^2}{1+\gamma\left|\zeta(k-d)+\zeta_e(k-d)\right|^2}\right] \times\frac{e_\epsilon^2(k)}{1+\gamma\left|\zeta(k-d)+\zeta_e(k-d)\right|^2}.$$

For $(V(k) - V(k-1)) \leq 0$, we need

$$\frac{-2}{\theta_1} + \frac{\eta\left|\zeta(k-d)+\zeta_e(k-d)\right|^2}{1+\gamma\left|\zeta(k-d)+\zeta_e(k-d)\right|^2} < 0. \tag{105}$$

Define $\Gamma(k)$ such that $\Gamma(k) = |\zeta(k) + \zeta_e(k)|^2$, where $\zeta(k)$ and $\zeta_e(k) \in \mathcal{R}^p$, and $\Gamma(k) \in \mathcal{R}^+$ ($\mathcal{R}^+$ is defined as the set of positive real numbers). It can be shown that

$$\Gamma(k) = |\zeta(k)|^2 + |\zeta_e(k)|^2 + 2\zeta(k)^\top\zeta_e(k). \tag{106}$$

Since $\zeta_e(k) = \Lambda(k)\zeta(k)$, (106) becomes

$$\begin{aligned} \Gamma(k) &= |\zeta(k)|^2 + |\Lambda(k)\zeta(k)|^2 + 2\zeta(k)^\top\Lambda(k)\zeta(k) \\ &= \sum_{i=1}^p \zeta_i^2(k) + \sum_{i=1}^p \Lambda_i^2(k)\zeta_i^2(k) + 2\sum_{i=1}^p \Lambda_i(k)\zeta_i^2(k) \\ &= \sum_{i=1}^p\left\{\zeta_i^2(k)\left[1+\Lambda_i^2(k)+2\Lambda_i(k)\right]\right\} \\ &= \sum_{i=1}^p\left\{\zeta_i^2(k)\left[1+\Lambda_i(k)\right]^2\right\}. \end{aligned} \tag{107}$$

Since $\Lambda_i(k) = 1 + \sigma_i(k)$, (107) becomes

$$\Gamma(k) = \sum_{i=1}^p\left\{\zeta_i^2(k)\left[\sigma_i(k)+2\right]^2\right\}. \tag{108}$$

Based on (105), to guarantee that $V(k) - V(k-d) \leq 0$, we need

$$\left[\frac{-2}{\theta_1} + \frac{\eta\Gamma(k-d)}{1+\gamma\Gamma(k-d)}\right] < 0 \tag{109}$$

or, equivalently

$$\frac{\eta\theta_1\Gamma(k-d) - 2\left(1+\gamma\Gamma(k-d)\right)}{\theta_1\left(1+\gamma\Gamma(k-d)\right)} < 0. \tag{110}$$

This can be achieved when

$$\Gamma(k-d)[\eta\theta_1 - 2\gamma] < 2. \tag{111}$$

Hence, we need to ensure that

$$\Gamma(k-d) < \frac{2}{\eta\theta_1 - 2\gamma}. \tag{112}$$

Combining (108) and (112), we need to guarantee that

$$\max_k\left\{\sum_{i=1}^p\left\{\zeta_i^2(k-d)\left[\sigma_i(k-d)+2\right]^2\right\}\right\} < \frac{2}{\eta\theta_1 - 2\gamma}. \tag{113}$$

Since $\sigma_i(k-d) \leq \bar{\sigma}$, we need to ensure that

$$\max_k\left\{[\bar{\sigma}+2]^2\sum_{i=1}^p\left\{\zeta_i^2(k-d)\right\}\right\} < \frac{2}{\eta\theta_1 - 2\gamma} \tag{114}$$

or (using Assumption 3)

$$[\bar{\sigma}+2]^2\zeta_{\max}^2 < \frac{2}{\eta\theta_1 - 2\gamma}. \tag{115}$$

It is clear from (115) that we need also to guarantee that $0 < \eta\theta_1 - 2\gamma$ (since both sides of the equation have to be positive). Therefore, choosing the parameters η and γ to satisfy

$$0 < (\eta\theta_1 - 2\gamma) < \frac{2}{(\bar{\sigma}+2)^2\zeta_{\max}^2} \tag{116}$$

will ensure that $V(k) - V(k-d) \leq 0$. ■

Note that the condition (101) may be infeasible for certain class of systems. However, if there is a class of systems for which this condition is satisfied, then stability can be guaranteed.

V. Surge Tank Example

Consider the surge tank model (taken from [21]) that can be represented by the following differential equation:

$$\frac{dh(t)}{dt} = \frac{-c\sqrt{2gh(t)}}{A_r(h(t))} + \frac{1}{A_r(h(t))}u(t) \tag{117}$$

where $u(t)$ is the input flow (control input), which can be positive or negative. Also, $h(t)$ is the liquid level (output of the system); $A_r(h(t))$ is the cross-sectional area of the tank; $g = 9.8\ \mathrm{m/sec}^2$ is the gravitational acceleration; and $c = 1$ is the known cross-sectional area of the output pipe. Let $A_r(h(t)) = \sqrt{ah(t)+b}$, where $a = 1$ and $b = 3$. Using Euler approximation to discretize the system, we have

$$h(k+1) = h(k) + T\left[\frac{-\sqrt{19.6h(k)}}{A_r(h(k))} + \frac{u(k)}{A_r(h(k))}\right] \tag{118}$$

where $T = 0.1$. Note that the system (118) belongs to the same class of systems (2), where $d = 1$

$$f_0(x(k)) = h(k) - \frac{T\sqrt{19.6h(k)}}{A_r(h(k))} \tag{119}$$

and

$$g_0(x(k)) = \frac{T}{A_r(h(k))}. \tag{120}$$

We will simulate the system for $h(k) > 0$ so that the simulation is realistic. This system will be used here to demonstrate how to use the previously discussed algorithms for auto-tuning the adaptation gain and the direction of descent. Next, auto-tuning the adaptation gain algorithm will be used for this example.

A. Auto-Tuning the Adaptation Gain

We have discussed earlier that auto-tuning the adaptation gain algorithm can be used for both direct and indirect adaptive control schemes. Our objective here is not only to discuss the applicability of using the proposed algorithm for both direct and indirect adaptive control schemes, but also to find a basis for comparison between the two adaptive control schemes within the context of the proposed algorithm.

1) Direct Case: The direct fuzzy controller $u(k) = A_u^\top \zeta(h(k), r(k))$ used here is a TSFS that has two inputs, the reference input (which is a square wave whose upper and lower values are 1.5 and 3) and error, $e(k) = r(k) - h(k)$. Five Gaussian membership functions are used for each input universe of discourse. For the first input (reference input), the centers of the membership functions are distributed evenly between 0 and 5, and the centers for the second input (error) are distributed evenly between -5 and 5. The choices of the parameters used in the update routine are made based on the following reasoning. Since the tank (118) belongs to the class of systems (2), the error dynamics can be expressed by (8) where

Fig. 3. Optimal adaptive direct fuzzy controller for the surge tank.

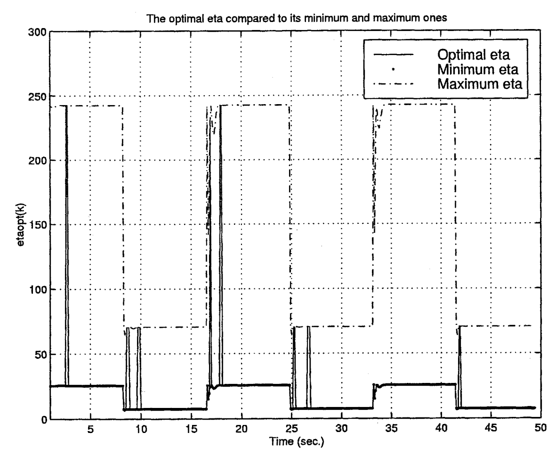

Fig. 4. Optimal adaptation gain compared to its upper and lower bounds.

$$\theta(x(k)) = g_0(x(k)) = \left[\frac{T}{A_r(h(k))}\right].$$

Since $\theta(x(k))$ must satisfy

$$0 < \theta_0 \leq \theta(x(k)) \leq \theta_1$$

and assuming that $0 < h(k) \leq 7$, it can be shown that the following relation must hold:

$$0.0316 \leq \theta(x(k)) \leq 0.0577$$

so let us pick θ_0 and θ_1 to be 0.033 and 0.05, respectively. It is easy to verify that this choice satisfies (32). Also, based on (41), $\alpha_2 - \alpha_1$ must be greater than 0.5152 so we can pick α_1 and α_2 to be 0.1 and 0.95, respectively. From (40), we know that ρ has to be less than 40 and also satisfy (39) (which says that ρ must lie in the closed interval [22.6061,36]) so one reasonable choice is to pick $\rho = 30$. The closed-loop response of the overall system is shown in Fig. 3. The first plot shows a comparison between the plant's output and the desired square wave reference trajectory. The second and third plots in the figure show the output error $e(k)$ and the control $u^{\mathrm{opt}}(k)$, respectively. A plot of the adaptation gain $\eta^{\mathrm{opt}}(k)$ (compared to the upper and lower bounds) is shown in Fig. 4. It is also clear from Fig. 4 that the lower bound

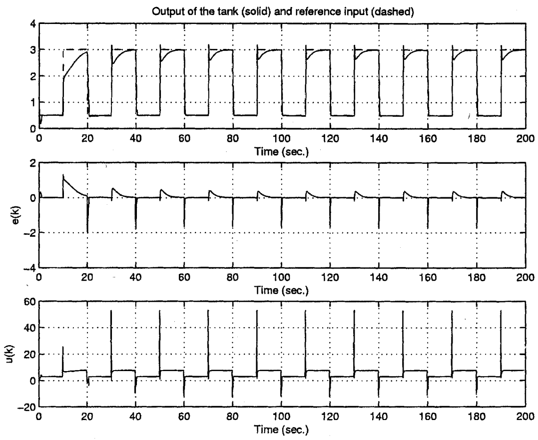

Fig. 5. Optimal indirect adaptive fuzzy controller for the surge tank.

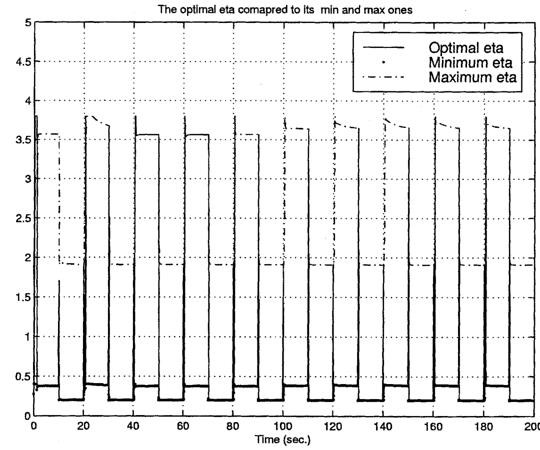

Fig. 6. Optimal adaptation gain compared to its upper and lower bounds.

of the adaptation gain $\eta_{\min}(k)$ is used as the optimal one when the output error is very small. This result agrees with our intuition since no major changes are needed in the adaptive controller when the achieved closed-loop performance is acceptable. In this case, no major changes in the parameter vector (and hence a small adaptation gain) are needed. In the case where the output error is fairly large, considerable changes in the adaptive controller are needed. This translates to major changes in the parameter vector (which, of course, require a large adaptation gain). As expected, the results provided by this example in the direct case verify our intuition on the choices of the adaptation gain.

2) Indirect Case: Here, two unknown functions ($f_0(x(k))$ and $g_0(x(k))$) are to be approximated online in order for the control law to be computed. The actual functions $f_0(x(k))$ and $g_0(x(k))$ (that we assume to be unknown) to be approximated are defined in (119) and (120), respectively. The approximators of both functions used here are TSFS that have one input, $h(k)$. Three Gaussian membership functions are used for the input in each approximator, where the centers are distributed evenly between 0 and 4. As in the direct case, we choose $\alpha_1 = 0.1$ and $\alpha_2 = 0.95$; this implies that $\rho = 1.5$. Also, to ensure that $\hat{g}_0(x(k))$ is bounded away from zero, we let $\hat{g}_0(x(k)) \geq \theta_0$ where $\theta_0 = 0.05$. To guarantee boundedness of the adaptation gain, we assumed that $|\zeta(k)|^2 \geq \bar{\zeta}$, where $\bar{\zeta} = 1$. Using $\gamma = 1$, this translates to having the upper bound on the maximum possible adaptation gain to be 4. The closed-loop response of the overall system is shown in Fig. 5. A plot of the optimal adaptation gain $\eta^{\rm opt}(k)$ (compared to the upper and lower bounds) is shown in Fig. 6. Also, in the second period of our simulation shown in Fig. 6 (where the parameter vector are converging toward the ideal ones), the lower bound of the adaptation gain is used as an optimal one. This agrees with our intuition as in the direct case. It is important to note that the convergence of the parameter vector in the indirect case in simulation is slower than the convergence in the direct case. This simulation result agrees with our conclusion which says that no matter what bounds we choose for our adaptation gain (as long as $0 < \alpha_1 < \alpha_2 < 1$), the parameter ρ is always bounded in the interval (0, 2); whereas in the direct case ρ can be much larger (in this case, we picked it to be 30).

B. Auto-Tuning the Direction of Descent

Here, we will demonstrate the impact of auto-tuning the direction of descent vector for the direct adaptive case; similar analysis can be performed also for the indirect adaptive control case.

The ideal controller is approximated here using a TSFS that has two inputs, the reference input (which is a square wave whose upper and lower values are 3 and 1.5) and error, $e(k) = r(k) - h(k)$. Five Gaussian membership functions are used for each input universe of discourse. For the first input (reference input), the centers of the membership functions are distributed evenly between 0 and 3, and the centers for the second input (error) are distributed evenly between -5 and 5. To satisfy Assumption 2, the function (91) is minimized subject to $-\bar{\sigma}\ \mathrm{abs}[\zeta(k)] \leq \zeta_e(k) \leq \bar{\sigma}\ \mathrm{abs}[\zeta(k)]$, where we choose $\bar{\sigma}$ is to be 0.7. It is assumed that $0 < \theta_0 \leq \theta(x(k)) \leq \theta_1$, where θ_0 and θ_1 are 0.033 and 0.05, respectively. Also, based on Assumption 3 (i.e., $|\zeta(k)| \leq \zeta_{\max}$), we assume $\zeta_{\max}$ to be 2. To guarantee that $0 < (\eta\theta_1 - 2\gamma)$, we choose the parameters η and γ to be 1 and 0.01, respectively. To satisfy (101), we choose $\bar{\sigma}$ to be 0.7. Note that (101) can be satisfied for other values of $\bar{\sigma}$. To study the effect of the direction of descent adaptation, we compared the case where the direction of descent is adapted to the case where it is not. The performance of the closed-loop system for the first case is shown in Fig. 7, where the first plot shows a comparison between the plant's output and the desired square wave reference trajectory. The second and third plots in the figure show the output error $e(k)$ and the new control $u(k)$, respectively. The performance of the closed-loop system for the second case (when no direction of descent adaptation is used) is shown in Fig. 8, where the first plot shows a comparison between the plant's output and the desired square wave reference trajectory. Also, the second and third plots in the figure show the output error $e(k)$ and the control $u(k)$, respectively. It is clear from the first plot in Fig. 7 that the closed-loop performance seems unacceptable at the beginning; however, the performance starts to improve over time. Note that the magnitude of overshoot in Fig. 7 (when the direction of descent is adapted) decreases faster than the corresponding one in Fig. 8 (when the direction of descent is not adapted). However, the output error

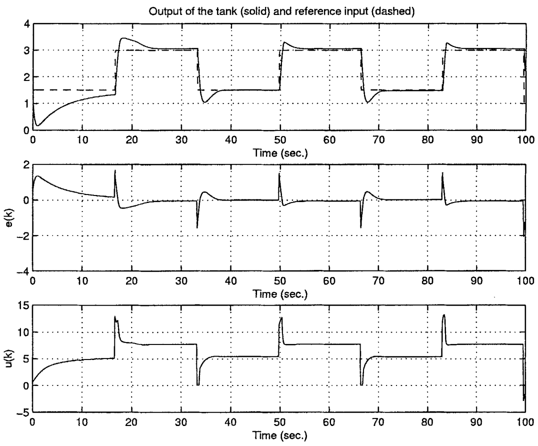

Fig. 7. Performance of direct adaptive controller when direction is adapted.

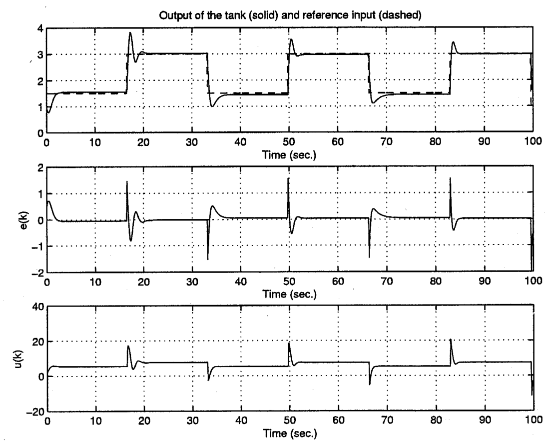

Fig. 8. Performance of direct adaptive controller when direction is not adapted.

in Fig. 8 (when the direction of descent is not adapted) decreases faster than the corresponding one in Fig. 7 (when the direction of descent is adapted). This observation agrees with our conclusions since the parameter vector in the first case is adapted to minimize only the squared output error; in the second case, however, there is a tradeoff between minimizing the squared instantaneous output error and the instantaneous control energy. Variations in the MCE and the MSE are shown in Fig. 9. The values shown in Fig. 9 are average values over a simulation period of 100 s. It is clear that the MCE decreases as $\bar{\sigma}$ increases. This observation makes sense since as $\bar{\sigma}$ increases, the incremental vector $\zeta_e(k)$ (which is found by minimizing the instantaneous control energy) may become larger, and hence $\zeta(k)$ may become greatly affected by $\zeta_e(k)$. Therefore, as $\bar{\sigma}$ increases the resulting $\zeta(k)$ will be obtained such that more consideration is given to minimizing the control energy. Also, in is clear from the figure that the MSE increases as $\bar{\sigma}$ increases. Hence, adapting the direction of descent can be used to tradeoff between the desired closed-loop performance relative to the affordable control energy.

VI. Concluding Remarks

Considering both direct and indirect adaptive control schemes, the main contribution of this paper is to auto-tune

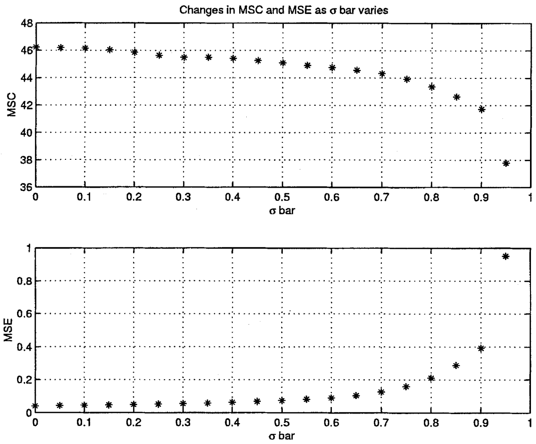

Fig. 9. Variations in MSE and MCE as $\bar{\sigma}$ varies.

some of the parameters (i.e., the adaptation gain and the direction of descent) for a gradient-based approximator parameter update law used for a class of nonlinear discrete-time systems. The adaptation mechanism of the gradient update law is usually based on minimizing the squared output error. Here, however, we update some parameters in the update law to minimize some other cost function (e.g., control energy). Based on the results of the example presented earlier, a comparison to some extent can be made between direct and indirect adaptive control schemes. Unlike the direct case, it is shown that the selection of the bounds of the adaptation gain in the indirect case is independent of the plant dynamics. This represents a major difference between the two adaptive control schemes. We have also found, from example, that auto-tuning the direction of descent helps to decrease the magnitude of overshoot and control energy. We were also able to conclude that auto-tuning the adaptation gain can be viewed as a special case of auto-tuning the direction of descent.

References

[1] K. Hornik, "Multilayered feedforward networks are universal approximators," *Neural Networks*, vol. 2, pp. 359–366, 1989.

[2] L.-X. Wang, "Fuzzy basis functions, universal approximation, and orthogonal least squares learning," *IEEE Trans. Neural Networks*, vol. 3, pp. 807–814, Oct. 1992.

[3] A. R. Barron, "Universal approximation bounds for superpositions of a sigmoid function," *IEEE Trans. Inform. Theory*, vol. 39, pp. 930–945, May 1993.

[4] J. L. Castro, "Fuzzy logic controllers are universal approximators," *IEEE Trans. Syst. Man, Cybern.*, vol. 25, pp. 624–635, Apr. 1995.

[5] Y. Song and J. Grizzle, "Adaptive output-feedback control of a class of discrete-time nonlinear systems," in *Proc. Amer. Control Conf.*, San Francisco, CA, June 1993, pp. 1359–1364.

[6] P.-C. Yeh and P. Kokotovic, "Adaptive output-feedback design for a class of nonlinear discrete-time systems," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 1663–1668, Sept. 1995.

[7] W. Lin and C. Byrnes, "h_∞-control of discrete time nonlinear systems," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 494–510, Apr. 1996.

[8] I. Kanellakopoulos, "A discrete time adaptive nonlinear system," in *Proc. 1994 Amer. Control Conf.*, Baltimore, MD, June 1994, pp. 867–869.

[9] ——, "A discrete time adaptive nonlinear system," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 2362–2365, Nov. 1994.

[10] A. Madani, S. Monaco, and D. Normand-Cyrot, "Adaptive control of discrete-time dynamics in parametric strict-feedback form," in *Proc. 35th Conf. Decision Control*, Kobe, Japan, Dec. 1996, pp. 2659–2664.

[11] F.-C. Chen and H. K. Khalil, "Adaptive control of nonlinear system using neural networks," *Int. J. Control*, vol. 55, no. 6, pp. 1299–1317, 1992.

[12] ——, "Adaptive control of a class of nonlinear discrete-time systems using neural networks," *IEEE Trans. Automat. Contr.*, vol. 40, pp. 791–801, May 1995.

[13] S. Jagannathan and F. L. Lewis, "Discrete-time neural net controler for a class of nonlinear dynamical systems," *IEEE Trans. Automat. Contr.*, vol. 41, pp. 1693–1699, Nov. 1996.

[14] ——, "Multilayer discrete-time neural-net controller with guaranteed performance," *IEEE Trans. Neural Networks*, vol. 7, pp. 107–130, Jan. 1996.

[15] S. Jagannathan, "Adaptive fuzzy logic control of feedback linearizable discrete-time dynamical systems under persistence of excitation," *Automatica*, vol. 34, no. 11, pp. 1295–1310, 1998.

[16] T. A. Johansen, "Fuzzy model based control: stability, robustness, and performance issues," *IEEE Trans. Fuzzy Syst.*, vol. 2, pp. 221–234, Aug. 1994.

[17] J. T. Spooner, R. Ordonez, and K. M. Passino, "Indirect adaptive fuzzy control far a class of discrete time systems," in *Proc. Amer. Control Conf.*, Albuquerque, NM, June 4–6, 1997, pp. 3311–3315.

[18] ——, "Direct adaptive fuzzy control for a class of discrete time systems," in *Proc. Amer. Control Conf.*, Albuquerque, NM, June 4–6, 1997, pp. 1814–1818.

[19] H. N. Nounou and K. M. Passino, "Stable auto-tuning of the adaptation gain for direct adaptive control," in *Proc. Amer. Control Conf.*, Chicago, IL, June 2000, pp. 2154–2158.

[20] ——, "Stable auto-tuning of the adaptation gain for indirect adaptive control," in *Proc. Amer. Control Conf.*, Chicago, IL, June 2000, pp. 2159–2163.

[21] K. M. Passino and S. Yurkovich, *Fuzzy Control*. Menlo Park, CA: Addison-Wesley Longman, 1998.

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