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Stability of a One-Dimensional Discrete-Time Asynchronous Swarm

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***Abstract*—In this correspondence, we consider a discrete time one-dimensional asynchronous swarm. First, we describe the mathematical model for motions of the swarm members. Then, we analyze the stability properties of that model. The stability concept that we consider, which matches exactly with stability of equilibria in control theory, characterizes stability of a particular position (relative arrangement) of the swarm members. We call that position the *comfortable position* (with *comfortable intermember distances*). Our swarm model and stability analysis are different from other asynchronous swarm models considered in the literature. In particular, in our analysis we employ results on contractive mappings from the parallel and distributed computation literature. The application of these results to the swarm coordination problem is important by itself since they might prove useful also in n-dimensional swarms.**

***Index Terms*—Aggregation, asynchronous motion, cooperative coordination and control, multiagent systems, swarms.**

I. Introduction

Many social organizms aggregate in groups and have the ability to perform cooperative and coordinated behavior as a group, such as socially foraging for food or avoiding predators. It was shown by Grünbaum in [1] that such cooperative and coordinated behavior, or simply "swarming behavior," may have evolved due to some survival advantages it may provide. (See also other relevant work in [2].) The area of modeling and analysis of swarming behavior is an active research area that has become more important due to its potential use in many areas including optimization [3] and robotics [4], among others. In particular, principles developed from studying swarms in nature could be very useful in characterizing and analyzing mechanisms for cooperative control for groups of autonomous robots.

In [5] Jin *et al.* studied the stability properties of one-dimensional (1-D) and two-dimensional synchronized swarms. The stability of 1-D swarms is similar to the concept of "platoon" stability in automated highway systems and there has been a significant work in that area (e.g., see [6]–[9]).

In [10] and [11], we considered a biologically inspired n-dimensional continuous time synchronous swarm model based on artificial potentials and obtained results on cohesive swarm aggregation. In [12], the model in [10] and [11] was augmented with a term representing the environment and convergence to (divergence from) more favorable regions (unfavorable regions) was shown. In [13], we discussed a procedure based on the sliding mode control technique which can be used to implement engineering aggregating swarms which are moving according to models such as those considered in [10]–[12]. However, the procedure considered in [13] is more general and can be used in other contexts as well, including formation control. Results based on artificial potentials and virtual leaders and in nature similar to those in [10]–[12] were independently obtained by Leonard *et al.* [14], [15] for agents with point mass dynamics. In [15], they also considered sampling affects on the swarm motion and gradient descent. Recently, Liu and Passino [16] obtained stability results of social foraging swarms moving in a noisy environment. They determined conditions under which the social foragers, modeled as point mass particles, will stay cohesive despite the noise present in the environment.

The above-mentioned articles mostly consider continuous time and/or synchronous swarm models. Beni and Liang [17] is, to best of our knowledge, one of the first stability results for asynchronous methods. There the authors consider a "linear" swarm model and prove sufficient conditions for the asynchronous convergence of the swarm to a synchronously achievable configuration. Although their method is asynchronous, they do not have time delays in the system. The stability of totally asynchronous swarm models (i.e., asynchronous swarm models with time delay) was, to best of our knowledge, first considered by Liu *et al.* in [18] and [19]. In [18], the authors consider 1-D discrete time totally asynchronous models for both stationary and mobile swarms and prove asymptotic convergence under total asynchronism conditions and finite time convergence under partial asynchronism conditions (i.e., total asynchronism with a bound on the maximum possible time delay). For the mobile swarm case they prove that cohesion will be preserved during motion under conditions expressed as bounds on the maximum possible time delay. In [19], the work in [18] has been extended to the multidimensional case by imposing special constraints on the "leader" movements and using a specific communication topology.

In this brief correspondence, we use the representation of a single swarm member considered by Liu *et al.* [18]; however, we consider a different mathematical model for the intermember interactions and motions in the swarm. In [18], the swarm members adjusted their position based on a single neighbor. This resulted in the fact that all the members must know the "comfortable interindividual distance" to which they intend to converge. This is not biologically very realistic since it requires agreement on that information between the members. In this correspondence, each member adjusts its position based on both of its neighbors and the "comfortable interindividual distance" is decided by only one member (which is the last member). Moreover, in this correspondence, we prove stability of the comfortable position for the new model using different mathematical tools for analysis (compared to these used in [18]). Namely, we use some earlier results developed for parallel and distributed computation in [20]. First, we prove stability for case of synchronism with no delays. Then, we use this result to prove stability under total asynchronism when there are also communication or sensing delays. The application of the results in [20] to the swarming problem is by itself important. This is because it might be possible to use them in problems such as stability of n-dimensional totally asynchronous swarms with time delays and for synchronization in the n-dimensional space. An initial version of the current correspondence appeared in [21].

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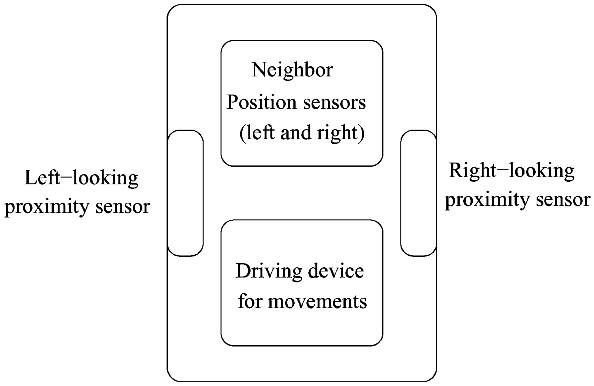

Fig. 1. Single swarm member.

II. Swarm Model

In this section, we introduce the swarm model that we use in this correspondence. First, we describe the model of a single swarm member. Then, we present the 1-D swarm model (i.e., when many swarm members are arranged next to each other on a line).

A. Single Swarm Member Model

The single swarm member model described in this section is taken from [18]. We present it here for convenience. The single swarm member model that we consider is shown in Fig. 1. As seen in the figure, it has a *driving device* for performing the movements and a *neighbor position sensors* for sensing the position of the adjacent (left and right) neighbors. It is assumed that there is no restriction on the range of these sensors. In other words, we assume that they can provide the accurate position of the neighbor even if the neighbor is far away. Each swarm member also has two proximity sensors on both sides (left and right). These sensors have sensing range of $\epsilon > 0$ and can sense *instantaneously* in this proximity. Therefore, if another swarm member reaches an ϵ distance from it, then this will be *instantaneously* known by both of the members. However, if the neighbors of the swarm member are out of the range of the proximity sensor, then it will return an infinite value (i.e., $-\infty$ for the left sensor and $+\infty$ for the right sensor) or some large number that will be ignored by the swarm member. The use of this sensor is to avoid collisions with the other members in the swarm.

In the next section, we describe the model of a swarm (i.e., a collection) of members described in this section arranged on a line.

B. One-Dimensional Swarm Model

Consider a discrete time 1-D swarm described by the model

$$\begin{aligned} x_1(k+1) &= x_1(k), \forall k \\ x_i(k+1) &= \max\{x_{i-1}(k)+\epsilon, \min\{p_i(k), x_{i+1}(k)-\epsilon\}\} \\ &\qquad \forall k \in \mathcal{K}^i, \quad i = 2,\ldots,N-1 \\ x_N(k+1) &= \max\{x_{N-1}(k)+\epsilon, p_N(k)\}, \quad \forall k \in \mathcal{K}^N \end{aligned} \tag{1}$$

where $x_i(k), i = 1,\ldots,N$, represents the position of member i at time k and $\mathcal{K}^i \subseteq \mathcal{K} = \{1,2,\ldots\}$ is the set of time instants at which member i updates its position. At the other time instants member i is stationary. In other words, we have

$$x_i(k+1) = x_i(k), \quad \forall k \notin \mathcal{K}^i \quad \text{and} \quad i = 2,\ldots,N. \tag{2}$$

The variables $p_i(k), i = 1,\ldots,N$, represent the *intended next positions* of the members and are given by

$$p_i(k) = x_i(k) - g(x_i(k) - c_i(k)), \qquad i = 2,\ldots,N$$

where $g(\cdot)$ is an attraction/repulsion function (to be discussed below). If there are not collision situations, then individual i will move to $p_i(k)$, otherwise it will stop at the safe distance ϵ form its neighbor. The quantities $c_i(k)$ represent the *perceived centers* of the adjacent neighbors of individual i. In other words, we have

$$c_i(k) = \frac{1}{2}\left[x_{i-1}\left(\tau_{i-1}^i(k)\right) + x_{i+1}\left(\tau_{i+1}^i(k)\right)\right]$$

for $i = 2,\ldots,N-1$, and

$$c_N(k) = x_{N-1}\left(\tau_{N-1}^N(k)\right) + d$$

where the constant d represents the *comfortable intermember distance*. The variable $\tau_j^i(k), j = i-1, i+1$, is used to represent the time index at which member i obtained position information of its neighbor j. It satisfies $0 \le \tau_j^i(k) \le k$ for $k \in \mathcal{K}^i$, where $\tau_j^i(k) = 0$ means that member i did not obtain any position information about member j so far (it still has the initial position information), whereas $\tau_j^i(k) = k$ means that it has the current position information of member j. The difference $(k - \tau_j^i(k)) \ge 0$ can be viewed as a sensing delay or a communication delay in obtaining the position information of agent j by agent i.

Note that in the swarm model in (1) it is implicitly assumed that $x_{i+1}(k) - x_{i-1}(k) > 2\epsilon$. Later we will show that this always will be the case provided that $x_{i+1}(0) - x_{i-1}(0) > 2\epsilon$ (which is satisfied by assumption). Also, notice that the first member of the swarm is always stationary at position $x_1(0)$. The other members (except member N), on the other hand, try to move to the position which their current information tells them is the middle of their adjacent neighbors. Of course due to the delays $c_i(k)$ may not be the midpoint between members $i-1$ and $i+1$ at time k. The last member (member N), on the other hand, tries to move to what it perceives to be a (comfortable) distance d from its left neighbor. Note that, in contrast to the work in [18], only the Nth member of the swarm knows (or decides on) the value of d. This is more advantageous since it does not require the achievement of agreement between the individuals (by negotiation or other means) on the same value of d. It is assumed that $d \gg \epsilon$, where the constant ϵ is the range of the proximity sensors as discussed in the preceding section. In [18], the authors also considered moving swarms and proved cohesiveness results assuming *partial asynchronism* and the existence of a bound on the maximum step size. However, here we will limit our analysis to the case of stationary swarms.

The elements of $\mathcal{K}$ (and therefore of $\mathcal{K}^i$) should be viewed as indices of the sequence of physical times at which the updates occur (similar to the times of events in discrete event systems), not as actual times. In other words, they are integers that can be mapped to actual times. The sets $\mathcal{K}^i$ are independent from each other for different i. However, it is possible to have $\mathcal{K}^i \cap \mathcal{K}^j \neq \emptyset$ for $i \neq j$ (i.e., two or more members may sometimes move simultaneously).

The function $g(\cdot)$ describes the attractive and repelling relationships between a swarm member and its adjacent neighbors. It determines the step size that a member will take toward the middle of its neighbors (if it is not already there). We assume that $g(\cdot)$ is sector bounded

$$\underline{\alpha} y^2 \le y g(y) \le \bar{\alpha} y^2 \tag{3}$$

where $\underline{\alpha}$ and $\bar{\alpha}$ are two constants satisfying

$$0 < \underline{\alpha} < \bar{\alpha} < 1.$$

Fig. 2 shows the plot of one such $g(\cdot)$. In the figure we also plotted $\underline{\alpha} y(t)$ and $\bar{\alpha} y(t)$ for $\underline{\alpha} = 0.1$ and $\bar{\alpha} = 0.9$.

Notice that the model in (1) is in a sense a discrete event model which does not allow for collisions between the swarm members. This is because if during movement member i suddenly finds itself within

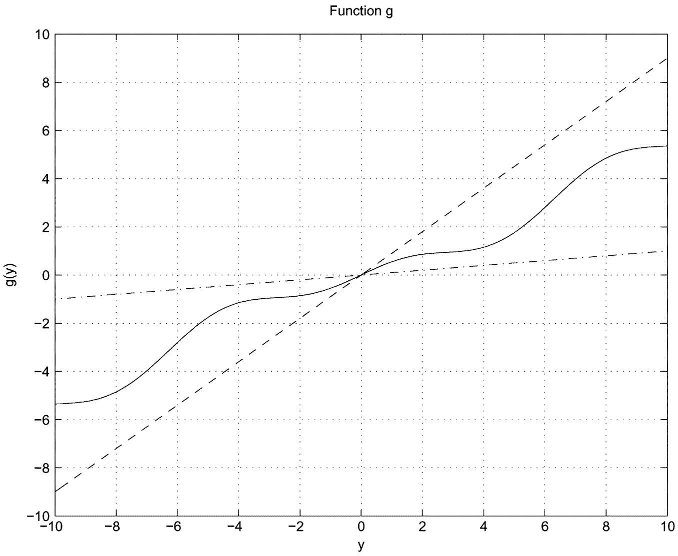

Fig. 2. Example $g(\cdot)$ function together with $0.1y(t)$ and $0.9y(t)$.

an ϵ range of one (or both) of its neighbors, it will restrain its movement toward that neighbor according to (1).

We will use the notation $x(k) = [x_1(k), \ldots, x_N(k)]^\top$ to represent the position at time k of all the members of the swarm. Define the swarm comfortable position as

$$x^c = [x_1(0), x_1(0) + d, \ldots, x_1(0) + (N-1)d]^\top.$$

In this correspondence, we consider the stability of this position by considering the motions of the swarm members when they are initialized at positions different from x^c. We will consider two cases: synchronous operation with no delays and totally asynchronous operation. These are described in the following two assumptions.

Assumption 1 (Synchronism, no Delays): The sets $\mathcal{K}^i$ and the times $\tau_j^i(k)$ satisfy $\mathcal{K}^i = \mathcal{K}$ for all i and $\tau_j^i(k) = k$ for all i and $j = i-1, i+1$.

This assumption states that all the swarm members will move at the same time instants. Moreover, every member will always have the current position information of its adjacent neighbors.

The next assumption, on the other hand, allows the members to move at totally independent time instants. Moreover, it allows the "delay" between two measurements performed by an individual to become arbitrarily large. However, it also states that there always will be a next time when the member will perform a measurement.

Assumption 2 (Total Asynchronism): The sets $\mathcal{K}^i$ are infinite, and if $\{k_\ell\}$ is a sequence of elements of $\mathcal{K}^i$ that tends to infinity, then $\lim_{\ell\to\infty} \tau_j^i(k_\ell) = \infty$ for every j.

Below we state a preliminary result about the swarm described by (1). We state it here, because it will be used in the next section.

Lemma 1: For the swarm described in (1) if Assumption 2 holds, then given any $x(0)$, there exists a constant $\bar{b} = \bar{b}(x(0))$ such that $x_i(k) \le \bar{b}$, for all k and all i, $1 \le i \le N$.

Proof: We prove this via contradiction. Assume that $x_{i+1}(k) \to \infty$ for some $i+1$, $1 \le i \le N$. This implies that $x_j(k) \to \infty$ for all $j \ge i+1$. We will show that it must be the case that $x_i(k) \to \infty$. Assume the contrary, i.e., assume that $x_{i+1}(k) \to \infty$, while $x_i(k) \le b < \infty$ for some b and for all k. Then we have $x_{i+1}(k) - x_i(k) \to \infty$, whereas $x_i(k) - x_{i-1}(k) < 2b_1$ for some $b_1 < b/2$. Let b_2 be a constant such that $(1 + (1/\underline{\alpha}))b < b_2 < \infty$. Note that $b_2 > b_1$. From Assumption 2 there is always a time $k_1^i \in \mathcal{K}^i$ at which member i performs position sensing of its neighbors and $\tau_{i-1}^i(k) \ge k_1^i$ and $\tau_{i+1}^i(k) \ge k_1^i$ for $k \ge k_1^i$ and

$$x_i(k) - x_{i-1}\left(\tau_{i-1}^i(k)\right) < 2b_1$$
$$x_{i+1}\left(\tau_{i+1}^i(k)\right) - x_i(k) > 2b_2.$$

This implies that we have $x_i(k) - c_i(k) < -(b_2 - b_1) < 0$. There exists also a time $k_2^i \ge k_1^i$ at which member i moves to the right and its new position satisfies

$$\begin{aligned} x_i\left(k_2^i + 1\right) &= x_i\left(k_2^i\right) - g\left(x_i\left(k_2^i\right) - c_i(k_2^i)\right) \\ &\ge x_i(k_2^i) - \underline{\alpha}\left(x_i\left(k_2^i\right) - c_i\left(k_2^i\right)\right) \\ &> x_i\left(k_2^i\right) + \underline{\alpha}(b_2 - b_1) \\ &> x_i\left(k_2^i\right) + b > b. \end{aligned}$$

This contradicts the assumption that $x_i(k) \le b$ for all k and implies that $x_i(k) \to \infty$ as well. Repeating the argument for the other members one obtains that $x_i(k) \to \infty$ for all $i \ne 1$.

Since $x_1 = x_1(0)$ for all k the above implies that $x_2(k) - x_1(k) \to \infty$ as $k \to \infty$. Moreover, it must be the case that $x_i(k) - x_{i-1}(k) \to \infty$ for all $i = 2, \ldots, N$. To see this assume that $x_i(k) - x_{i-1}(k) \to \infty$, whereas $x_{i+1}(k) - x_i(k) < b_1$ for some b_1. There exist a time $k_3^i \in \mathcal{K}^i$ at which member i performs position sensing of its neighbors and $\tau_{i-1}^i(k) \ge k_3^i$ and $\tau_{i+1}^i(k) \ge k_3^i$ for $k \ge k_3^i$. Moreover, for $k \ge k_3^i$ we have

$$x_i(k) - x_{i-1}\left(\tau_{i-1}^i(k)\right) > 2b_2$$
$$x_{i+1}\left(\tau_{i+1}^i(k)\right) - x_i(k) < 2b_1$$

and $x_i(k) < b$ for some $b > b_2$. This implies that we have $x_i(k) - c_i(k) > (b_2 - b_1) > 0$. There exists also a time $k_4^i \ge k_3^i$ at which member i moves to the left and its new position satisfies

$$\begin{aligned} x_i(k_4^i + 1) &= x_i\left(k_4^i\right) - g\left(x_i\left(k_4^i\right) - c_i\left(k_4^i\right)\right) \\ &\le x_i\left(k_4^i\right) - \underline{\alpha}\left(x_i\left(k_4^i\right) - c_i\left(k_4^i\right)\right) \\ &< x_i\left(k_4^i\right) - \underline{\alpha}(b_2 - b_1) \\ &< x_i\left(k_4^i\right) < b. \end{aligned}$$

Note that as long as we have $x_i(k) - c_i(k) > (b_2 - b_1) > 0$ the agent will be moving to the left and $x_i(k)$ will always be bounded by b. Therefore, it cannot be the case that $x_{i+1}(k) - x_i(k) < b_1$, while $x_i(k) - x_{i-1}(k) \to \infty$. Therefore, if $x_i(k) \to \infty$ for some $i > 1$, then it must be the case that $x_i(k) \to \infty$ and $x_i(k) - x_{i-1}(k) \to \infty$ for all $i = 2, \ldots, N$.

Take individual N. From above we know that there exists a time k_5^N such that $x_N(k) - x_{N-1}(k) > d$. However, there is always a time $k_6^N > k_5^N$ such that member N performs position sensing and $\tau_{N-1}^N(k) \ge k_6^N$ for $k \ge k_6^N$. Then, we have $x_N(k) - x_{N-1}(\tau_{N-1}^N(k)) > d$ and at some time $k_7^N > k_6^N$ the member moves and

$$x_N\left(k_7^N + 1\right) = x_N\left(k_7^N\right) - g\left(x_N\left(k_7^N\right) - c_N\left(k_7^N\right)\right) < x_N\left(k_7^N\right)$$

implying that it moves to the left. In fact, as long as we have $x_N(k) - x_{N-1}(\tau_{N-1}^N(k)) > d$ individual N moves to the left and it cannot diverge far away from its neighbor. Assuming that at time k_6^N we have

$x_N(k) - x_{N-1}(k) < b$ for some $b > d$, this implies that $x_N(k) - x_{N-1}(k) < b$ for all $k > k_6^N$. Therefore, the Nth individual cannot diverge leading to a contradiction. ■

This result is important, because it basically says that for the given swarm model both swarm member positions and intermember distances are bounded (implying that the swarm will not dissolve) despite the asynchronism and the time delays. Therefore, the main question to be answered is whether the swarm member positions $x(k)$ will converge to some constant, will have periodic solutions or will exhibit chaotic behavior. In the next section we will analyze the system in the case of synchronism with no delays. This will be used later in the proof of our main result.

III. System Under Total Synchronism

In this section we will assume that Assumption 1 holds (i.e., all the members move at the same time and they always have the current position information of their neighbors) and analyze the stability properties of the system.

First, we state the following preliminary result.

Lemma 2: For the system in (1) assume that Assumption 1 holds (i.e., we have synchronism with no delays). If $x(k) \to \bar{x}$ as $k \to \infty$, where $\bar{x}$ is a constant vector, then $\bar{x} = x^c$.

Proof: First of all, note that the intermember distances on all the states that the system can converge to are such that $\bar{x}_i - \bar{x}_{i-1} > \epsilon$ for all i (i.e., it is impossible for the states to converge to positions that are very close to each other). To prove this, we assume that $\bar{x}_i - \bar{x}_{i-1} = \epsilon$ for some i and $\bar{x}_j - \bar{x}_{j-1} > \epsilon$ for all $j \neq i$ and seek to show a contradiction. In that case, $\bar{x}_{i+1} - \bar{x}_i > \epsilon$ so

$$\bar{x}_i - \bar{c}_i = \bar{x}_i - \frac{\bar{x}_{i-1} + \bar{x}_{i+1}}{2} < 0$$

and we have from model constraints in (1) that

$$\bar{x}_{i-1} + \epsilon < \bar{x}_i - g\left(\bar{x}_i - \frac{\bar{x}_{i-1} + \bar{x}_{i+1}}{2}\right) < \bar{x}_{i+1} - \epsilon.$$

From (1) this implies that at the next time instant $k^i \in \mathcal{K}^i$ member i will move to the right toward member $i+1$. Therefore, it must be the case that $\bar{x}_{i+1} - \bar{x}_i = \epsilon$ since otherwise $\bar{x}_i - \bar{x}_{i-1} = \epsilon$ also cannot hold. Continuing this way one can prove that all intermember distances must be equal to ϵ. However, in that case, since $d \gg \epsilon$, from last equality in (1) we have

$$\bar{x}_N - g(\epsilon - d) > \bar{x}_{N-1} + \epsilon$$

and this implies that on the next time instant $k^N \in \mathcal{K}^N$ member N will move to the right. Therefore, no intermember distance can converge to ϵ. For this reason, to find $\bar{x}$ we can drop the min and max and consider only the middle terms in (1).

Since $x(k) \to \bar{x}$ as $t \to \infty$ it should be the case that ultimately

$$\begin{aligned} \bar{x}_1 &= \bar{x}_1 \\ \bar{x}_i &= \bar{x}_i - g\left(\bar{x}_i - \frac{\bar{x}_{i-1} + \bar{x}_{i+1}}{2}\right), \qquad i = 1, \ldots, N-1 \\ \bar{x}_N &= \bar{x}_N - g(\bar{x}_N - \bar{x}_{N-1} - d) \end{aligned}$$

from which we obtain

$$\begin{aligned} \bar{x}_1 &= x_1^c \\ 2\bar{x}_i &= \bar{x}_{i-1} + \bar{x}_{i+1}, \qquad i = 1, \ldots, N-1 \\ \bar{x}_N &= \bar{x}_{N-1} + d. \end{aligned} \tag{4}$$

Solving the second equation for $\bar{x}_{N-1}$ we have $2\bar{x}_{N-1} = \bar{x}_{N-2} + \bar{x}_N$ from which we obtain $\bar{x}_{N-1} = \bar{x}_{N-2} + d$. Continuing this way, we obtain

$$\bar{x}_i = \bar{x}_{i-1} + d$$

for all $i = 1, \ldots, N-1$. Then since the first member is stationary we have $\bar{x}_1 = x_1(t) = x_1(0) = x_1^c$ and this proves the result. ■

The implication of this lemma is basically that x^c is the unique *fixed point* or *equilibrium point* of the system described by (1). In this correspondence, we analyze the stability of this fixed point, which corresponds to the arrangement with comfortable intermember distance.

Lemma 3: Assume that $x_i(0) - x_{i-1}(0) > \epsilon$ for all $i = 2, \ldots, N$. Moreover, assume that Assumption 1 holds (i.e., we have synchronism with no delays). Then, $x_i(k) - x_{i-1}(k) > \epsilon$ for all $i = 2, \ldots, N$, and for all k.

Proof: We will prove this by induction. By assumption for $k = 0$ we have $x_i(0) - x_{i-1}(0) > \epsilon$ for all $i = 2, \ldots, N$. Assume that for some k we have $x_i(k) - x_{i-1}(k) > \epsilon$ for all $i = 2, \ldots, N$. Then, with a simple manipulation one can show that at that time k we have

$$x_{i-1}(k) + \epsilon < x_i(k) < x_{i+1}(k) - \epsilon$$

$$x_{i-1}(k) + \epsilon < c_i(k) < x_{i+1}(k) - \epsilon \tag{5}$$

$$c_{i+1}(k) - c_i(k) > \epsilon \tag{6}$$

for all $i = 2, \ldots, N$.

Noting that it is possible to write the $g(\cdot)$ function as

$$g(y(k)) = \alpha(k) y(k)$$

where $0 < \underline{\alpha} \leq \alpha(k) \leq \bar{\alpha} < 1$, and using this in the swarm dynamics equation we have

$$\begin{aligned} x_i(k+1) &= x_i(k) - \alpha_i(k)(x_i(k) - c_i(k)) \\ &= (1 - \alpha_i(k))x_i(k) + \alpha_i(k)c_i(k). \end{aligned}$$

Therefore, we have

$$\begin{aligned} x_i(k) < c_i(k) &\Rightarrow x_i(k) < x_i(k+1) < c_i(k) \\ x_i(k) > c_i(k) &\Rightarrow x_i(k) > x_i(k+1) > c_i(k). \end{aligned}$$

These inequalities together with (5) and (6) imply that

$$x_i(k+1) - x_{i-1}(k+1) > \epsilon$$

and this completes the proof. ■

This lemma implies that for the synchronous case with no delays, provided that initially the members are sufficiently apart from each other, the proximity sensors will not be used and that we can drop the min and max operations in (1) and the system can be represented as

$$\begin{aligned} x_1(k+1) &= x_1(k) \\ x_i(k+1) &= x_i(k) - g\left(x_i(k) - \frac{x_{i-1}(k) + x_{i+1}(k)}{2}\right) \\ x_N(k+1) &= x_N(k) - g(x_N(k) - x_{N-1}(k) - d). \end{aligned}$$

Define the following change of coordinates

$$\begin{aligned} e_1(k) &= x_1(k) - x_1^c \\ e_i(k) &= x_i(k) - (x_{i-1}(k) + d), \qquad i = 2, \ldots, N. \end{aligned}$$

Then, one obtains the following representation of the system

$$\begin{aligned} e_1(k+1) &= e_1(k) = 0 \\ e_2(k+1) &= e_2(k) - g\left(\frac{e_2(k) - e_3(k)}{2}\right) \end{aligned}$$

$$e_i(k+1) = e_i(k) - g\left(\frac{e_i(k)-e_{i+1}(k)}{2}\right) + g\left(\frac{e_{i-1}(k)-e_i(k)}{2}\right), \qquad i=3,\ldots,N-1$$
$$e_N(k+1) = e_N(k) - g(e_N(k)) + g\left(\frac{e_{N-1}(k)-e_N(k)}{2}\right).$$

Using again the fact that $g(y(k)) = \alpha(k)y(k)$ for $0 < \underline{\alpha} < \alpha(k) < \bar{\alpha} < 1$ we can represent the system with

$$e_2(k+1) = \left(1-\frac{\alpha_2(k)}{2}\right)e_2(k) + \frac{\alpha_2(k)}{2}e_3(k)$$
$$e_i(k+1) = \left(1-\frac{\alpha_i(k)}{2}-\frac{\alpha_{i-1}(k)}{2}\right)e_i(k) + \frac{\alpha_{i-1}(k)}{2}e_{i-1}(k) + \frac{\alpha_i(k)}{2}e_{i+1}(k), \qquad i=3,\ldots,N-1$$
$$e_N(k+1) = \left(1-\alpha_N(k)-\frac{\alpha_{N-1}(k)}{2}\right)e_N(k) + \frac{\alpha_{N-1}(k)}{2}e_{N-1}(k)$$

where we dropped $e_1(k)$ since it is zero for all k. Defining $e(k) = [e_2(k),\ldots,e_N(k)]^\top$ it can be seen that our system is, in a sense, a linear time varying system, which can be represented in a matrix form as

$$e(k+1) = A(k)e(k)$$

where $A(k)$ is a symmetric tridiagonal matrix of the form

$$A(k) = \begin{bmatrix} b_1 & c_1 & 0 & \cdots & 0 \\ c_1 & b_2 & c_2 & \ddots & \vdots \\ 0 & c_2 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & b_{N-2} & c_{N-2} \\ 0 & \cdots & 0 & c_{N-2} & b_{N-1} \end{bmatrix}$$

with diagonal elements given by

$$\{b_1,\ldots,b_{N-1}\} = \left\{\left(1-\frac{\alpha_2(k)}{2}\right), \left(1-\frac{\alpha_3(k)}{2}-\frac{\alpha_2(k)}{2}\right), \ldots, \left(1-\frac{\alpha_{N-1}(k)}{2}-\frac{\alpha_{N-2}(k)}{2}\right) \left(1-\alpha_N(k)-\frac{\alpha_{N-1}(k)}{2}\right)\right\}$$

and offdiagonal elements equal to

$$\{c_1,\ldots,c_{N-2}\} = \left\{\frac{\alpha_2(k)}{2},\ldots,\frac{\alpha_{N-1}(k)}{2}\right\}.$$

First, we will investigate the properties of the matrix $A(k)$ which will be useful later in deriving our stability result. In particular, we will show that all the eigenvalues of $A(k)$ lie within the unit circle.

Lemma 4: The spectrum of the matrix $A(k), \rho(A(k))$ satisfies

$$\rho(A(k)) < 1$$

for all k. In other words, all the eigenvalues of $A(k)$ lie within the unit circle.

Proof: First, note that for the given $A(k)$ we have

$$\|A(k)\|_1 = \|A(k)\|_\infty = 1$$

for all k. On the other hand, for any given matrix $A(k)$ it is well known that the two norm satisfies

$$\|A(k)\|_2 \le \|A(k)\|_1 \|A(k)\|_\infty.$$

Hence, since we have $\rho(A(k)) = \|A(k)\|_2$, we obtain

$$\rho(A(k)) \le 1$$

for all k. Now we must show that this holds with strict inequality. To prove that let us first assume that

$$\underline{\alpha} \le \alpha_i(k) = \alpha_i \le \bar{\alpha}$$

for all k and $i = 2,\ldots,N$ (i.e., the α_i's in the matrix A are all constants implying that the function $g(\cdot)$ is linear). Note also that $A(k)$ is a symmetric matrix. Therefore, there exists a unitary transformation T (i.e., $T^{-1} = T^\top$) such that $\bar{A} = TAT^\top$, where $\bar{A} = \operatorname{diag}\{\bar{a}_2,\ldots,\bar{a}_N\}$. For the sake of contradiction assume that $\rho(A) = 1$. Then, it must be the case that $\bar{a}_i = 1$ for some $i, 2 \le i \le N$. Define the transformation $\bar{e} = Te$. Then the system can be described as

$$\bar{e}(k+1) = \bar{A}\bar{e}(k).$$

Since $\bar{A}$ is diagonal and $\bar{a}_i = 1$ we have $\bar{e}_i(k) = \bar{e}_i(0)$ for all k, whereas $\bar{e}_j(k) \to 0$ as $k \to \infty$ for all $j \ne i$. This, on the other hand, implies that $e(k) \to T_i\bar{e}_i(0) = e^c$ as $k \to \infty$, where T_i is the ith column of T. Depending on the value of $\bar{e}_i(0)$, the value of e^c can be any number. However, this contradicts the result of Lemma 2. Therefore, $\bar{a}_i < 1$ for all $i = 2,\ldots,N$, and this implies that $\rho(A) < 1$. Since $\alpha = [\alpha_2,\ldots,\alpha_N]^\top$ was chosen arbitrary, the result holds for all α such that $\underline{\alpha} \le \alpha_i \le \bar{\alpha}$. Hence, we have

$$\rho(A(k)) < 1$$

for each k. ■

This lemma basically states that the eigenvalues of $A(k)$ (which are all real numbers since $A(k)$ is symmetric) lie within the unit circle for each k. Note that the $A(k)$ matrices are irreducible, nonnegative, tridiagonal, and symmetric. Moreover, they have entries which are overbounded by the entries of corresponding stochastic matrices. Therefore, the above result can be also proved using the Perron–Frobenious theorem [22].

Before proceeding define

$$\bar{\rho} = \sup_{\underline{\alpha} \le \alpha_i \le \bar{\alpha}, i=2\ldots N} \{\rho(A)\}.$$

Then, from the above result we have

$$\bar{\rho} < 1.$$

Using this, one can state the following result for the system under total synchronism.

Theorem 1: For the N-member swarm modeled in (1) with $g(\cdot)$ as given in (3), if Assumption 1 holds and $x_{i+1}(0) - x_i(0) > \epsilon, i = 1,\ldots,N-1$, then we have $e(k) \to 0$ as $k \to \infty$.

Proof: Directly follows from the above since

$$\begin{aligned}\|e(k)\|_2 &= \|A(k-1)A(k-2)\ldots A(0)e(0)\|_2 \\ &\le \|A(k-1)\|_2\|A(k-2)\|_2\ldots\|A(0)\|_2\,\|e(0)\|_2 \\ &\le \bar{\rho}^k\|e(0)\|_2.\end{aligned}$$
■

This theorem implies that the swarm member positions will asymptotically converge to the comfortable position x^c. It is an important result; however, it is not the main result of this correspondence. Our objective is to prove that the same type of convergence will be achieved for the totally asynchronous case. We will investigate that case in the following section and the result of Theorem 1 will be useful in its proof.

IV. Main Result

In this section we return to the totally asynchronous case. In other words, we investigate the system under Assumption 2. To prove convergence to the comfortable position x^c we will use the result from the synchronous case and a result from [20]. For convenience we present this result here.

Consider the function $f : X \to X$, where $X = X_1 \times \cdots \times X_n$, and $x = [x_1, \ldots, x_n]^\top$ with $x_i \in X_i$. The function f is composed of functions $f_i : X \to X_i$ in the form $f = [f_1, \ldots, f_n]^\top$ for all $x \in X$. Consider the problem of finding the point x^* such that

$$x^* = f(x^*)$$

using an asynchronous algorithm. In other words, use an algorithm in which

$$x_i(k+1) = f_i\left(x_1\left(\tau_1^i(k)\right), \ldots, x_n\left(\tau_n^i(k)\right)\right), \quad \forall k \in \mathcal{K}^i \quad (7)$$

where $\tau_j^i(k)$ are times satisfying $0 \le \tau_j^i(k) \le k, \forall k \in \mathcal{K}$. For all the other times $k \notin \mathcal{K}^i$, x_i is left unchanged. In other words, we have

$$x_i(k+1) = x_i(k), \quad \forall k \notin \mathcal{K}^i. \quad (8)$$

Consider the following assumption.

Assumption 3: There is a sequence of nonempty sets $\{X(k)\}$ with

$$\cdots \subset X(k+1) \subset X(k) \subset \cdots \subset X$$

satisfying the following two conditions:

1) *Synchronous Convergence Condition (SCC)*: We have

$$f(x) \in X(k+1), \quad \forall k \text{ and } x \in X(k).$$

Furthermore, if $\{y_k \mid y_k = f(y_{k-1})\}$ is a sequence such that $y_k \in X(k)$ for every k, then every limit point of $\{y_k\}$ is a fixed point of f.

2) *Box Condition (BC)*: For every k, there exist sets $X_i(k) \subset X_i$ such that

$$X(k) = X_1(k) \times X_2(k) \times \ldots X_n(k).$$

The SCC condition implies that the limit points of the sequences generated by the *synchronous iteration* $x(k+1) = f(x(k))$ are fixed points of f. The BC condition, on the other hand, implies that combining components of vectors in $X(k)$ results in a vector in $X(k)$. In other words, if $x \in X(k)$ and $\bar{x} \in X(k)$, then replacing ith component of x with the ith component of $\bar{x}$ results in a vector in $X(k)$. An example when BC holds is when $X(k)$ is sphere in $\mathcal{R}^n$ with respect to some weighted maximum norm.

Assumption 3 is about the convergence of the synchronous iteration (i.e., iteration (7) under total synchronism). The result below shows that if the synchronous algorithm is convergent, the asynchronous algorithm will also converge provided that Assumption 2 is satisfied.

Theorem 2: Asynchronous Convergence Theorem [20]: If the Synchronous Convergence Condition and Box Condition of Assumption 3 hold together with Assumption 2, and the initial solution estimate $x(0) = [x_1(0), \ldots, x_n(0)]^\top$ belongs to the set $X(0)$, then every limit point of $\{x(k)\}$ is a fixed point of f.

This is a powerful result that can be applied to many different problems. The main idea behind its proof is as follows. Assume there is a time instant k_1 such that $x_j(\tau_j^i(k_1)) \in X_j(k)$ for all j and all i, which implies that *the perceived* $x(k_1)$ is in $X(k)$. Then the SCC condition in Assumption 3 together with the iteration in (7) guarantee that $x(k_1+1) \in X(k+1)$, whereas the BC condition implies that each $x_j(k_1+1) \in X_j(k+1)$. Also it is guaranteed that we have $x(k) \in X(k+1)$ for all $k \ge k_1$. Then, due to the total asynchronism assumption (Assumption 2) there will be always another time instant $k_2 > k_1$ such that $x_j(\tau_j^i(k_2)) \in X_j(k+1)$ for all j and all i. Repeating the above argument we have $x(k_2+1) \in X(k+2)$ and this completes the induction step since initially we have $x_j(\tau_j^i(0)) = x_j(0) \in X_j(0)$.

We will use the above theorem to prove our main result, which is as follows.

Theorem 3: For the N-member swarm modeled in (1) with $g(\cdot)$ as given in (3), if Assumption 2 holds and $x_{i+1}(0) - x_i(0) > \epsilon, i = 1, \ldots, N-1$, then the swarm member positions will converge asymptotically to the comfortable position x^c.

Proof: In order to prove this result we once again consider the synchronous case. Recall that for this case the system can be described by

$$e(k+1) = A(k)e(k).$$

In the previous section it was shown that for the synchronous case we have $\lambda(A(k)) \le \bar{\rho} < 1$ for all k and that $e(k) \to 0$ as $k \to \infty$. This implies that $A(k)$ is a maximum norm contraction mapping for all k. Define the sets

$$E(k) = \{e \in \mathbb{R}^{N-1} : \|e\|_\infty \le \bar{\rho}^k \|e(0)\|_\infty\}.$$

Then since $A(k)$ is a maximum norm contraction mapping for all k we have $e(k) \in E(k)$ for all k and

$$\ldots \subset E(k+1) \subset E(k) \subset \cdots \subset E = \mathbb{R}^{N-1}.$$

Moreover, each $E(k)$ can be expressed as

$$E(k) = E_2(k) \times E_3(k) \times \cdots E_N(k)$$

where $E_i(k)$ is such that $e_i(k) \in E_i(k)$. Since the position with comfortable intermember distance $e = 0$ (i.e., $x = x^c$) is the unique fixed point of the system and the synchronous swarm converges to it, it is implied that Assumption 3 above is satisfied. Applying the Asynchronous Convergence Theorem we obtain the result. ■

This result is important because it states that the stability of the system will be preserved (i.e., the system will converge to the comfortable distance) even though we have totally asynchronous motions and imperfect information due to the time delays. Note that the fact that in the asynchronous case in (1) the $\min$ and $\max$ operations are preserved does not change the result in Theorem 3 since the stability properties of the synchronous system is preserved even with them (i.e., the $\min$ and $\max$ operations) present in the model.

A relevant issue to mention here is the speed of convergence of the algorithm. Theorem 3 does not provide any statement about the speed of convergence. In fact, it can be seen from Assumption 2 that it is not possible to establish a lower bound on the speed of convergence due to the asynchronism and the possibility of unbounded time delays. Therefore, the convergence speed may vary and sometimes may take very long. If a faster convergence is desired a bound on the time delay of the form $0 \le \tau_j^i(k) \le B$ for some $B > 0$ can be assumed and with that assumption it might be possible to establish a stronger result specifying an upper (lower) bound on the convergence time (speed). Such an algorithm is called *partially asynchronous*.

Note also that the swarm equation in (1) is naturally distributed, where each individual i performs the computation only of its next position $x_i(k+1)$ based on the perceived (measured or obtained by communication) position of its two neighbors. Moreover, it performs this computation only at the times it is "awake" (i.e., only at $k \in \mathcal{K}^i$). Therefore, the computational load of the individuals is minimal.

A direct consequence of Theorem 3 is the stability of a swarm in which one member in the middle is stationary, whereas all the other middle members try to move similar to the middle members in the model in (1) and both of the edge members try to move to a distance d from their neighbors. In other words, suppose the swarm is described by

$$\begin{aligned} x_1(k+1) &= \min\{p_1(k), x_2(k) - \epsilon\} \quad \forall k \in \mathcal{K}^1 \\ x_j(k+1) &= x_j(k), \quad \forall k \text{ and for some } j, \quad 1 \le j \le N \\ x_i(k+1) &= \max\{x_{i-1}(k) + \epsilon, \min\{p_i(k), x_{i+1}(k) - \epsilon\}\} \\ &\qquad \forall k \in \mathcal{K}^i, \quad i = 2, \ldots, N-1, \quad i \neq j \\ x_N(k+1) &= \max\{x_{N-1}(k) + \epsilon, p_N(k)\}, \quad \forall k \in \mathcal{K}^N \end{aligned} \tag{9}$$

where $p_i(k)$'s are as defined before and the perceived center for the first member is

$$c_1(k) = x_2\left(\tau_2^1(k)\right) - d.$$

Here again it is implicitly assumed that $x_{i+1}(k) - x_{i-1}(k) > 2\epsilon$, as was the case in the preceding section. Recall that this is always the case provided that $x_{i+1}(0) - x_{i-1}(0) > 2\epsilon$. In this case we have the following corollary as a direct consequence of Theorem 3.

Corollary 1: For the N-member swarm modeled in (9) with $g(\cdot)$ as given in (3), if Assumption (2) holds and $x_{i+1}(0) - x_i(0) > \epsilon, i = 1, \ldots, N-1$, then the swarm member positions will converge asymptotically to x^c, where x^c is defined such that $x_j^c = x_j(0)$ and $x_i^c = x_j(0) + (i-j)d$, for all $i \neq j$.

The importance of this result is for systems in which the "leader" of the swarm is not the first (or the last) member, but a member in the middle. It is also worth mentioning here that if the first and the last members employ two different desired interindividual distances d_1 and d_N, $d_1 \neq d_N$, then these will be the intermember distances on the corresponding sides of the stationary member j. This result is also directly implied by Theorem 3 and can be stated as another corollary. The proofs follow from considering each side of the stationary member separately and applying Theorem 3.

V. Simulation Results

In this section we provide numerical simulation examples. We chose $N = 6$ members and $d = 1$ as the desired comfortable distance. As an attraction/repulsion function we used the linear function $g(y) = \alpha y$ with $\alpha = 0.9$. To achieve asynchronism at each time step the swarm members are set up to sense their neighbor positions and to update their own position with some probability. In particular, we defined two threshold probabilities $0 < \bar{p}_{\text{sense}} < 1$ and $0 < \bar{p}_{\text{move}} < 1$. Each time instant k for each member i two random numbers $0 < p_{\text{sense}}^i(k) < 1$ and $0 < p_{\text{move}}^i(k) < 1$ are generated with uniform probability density. If $p_{\text{sense}}^i(k) > \bar{p}_{\text{sense}}$, then member i performs neighbor position sensing (i.e., obtains the current position of its neighbors). Otherwise, it keeps their old position information. Similarly, if $p_{\text{move}}^i(k) > \bar{p}_{\text{move}}$, then member i updates its position according to (1). Otherwise, it keeps its current position. In the simulations below we used $\bar{p}_{\text{sense}} = 0.9$ and $\bar{p}_{\text{move}} = 0.9$.

Fig. 3 shows a simulation of a contracting swarm (i.e., a swarm in which the members are far apart from each other initially). The members move to the comfortable position ($d = 1$ apart from each other) as time progresses, as expected.

Fig. 4 shows a simulation of an expanding swarm (i.e., a swarm in which the members are close to each other initially). In this simulation the results are also as the theory predicts and the swarm converges to the desired comfortable relative position.

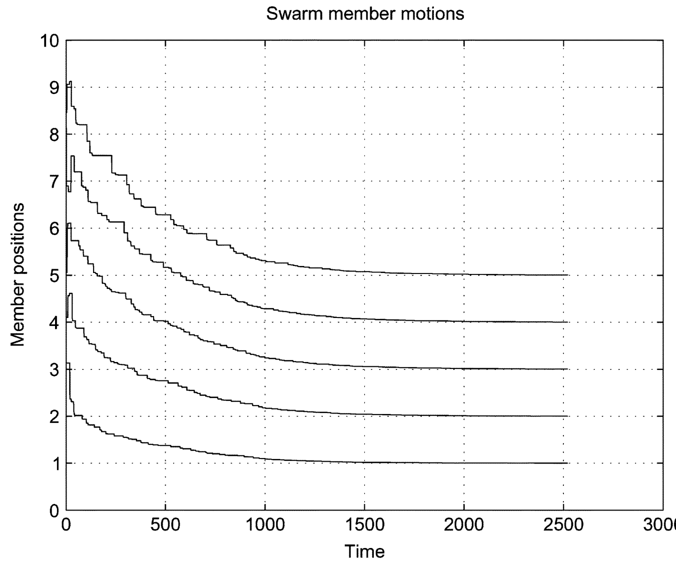

Fig. 3. Contracting swarm.

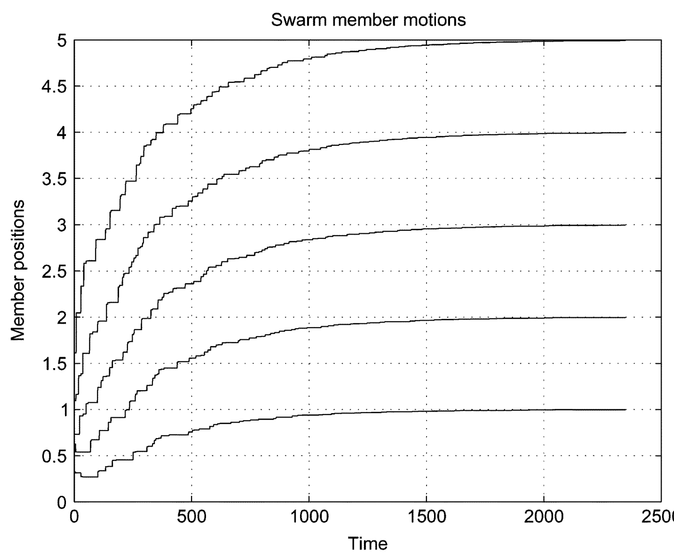

Fig. 4. Expanding swarm.

In the simulation by choosing the values of $\bar{p}_{\text{sense}}$ and $\bar{p}_{\text{move}}$ one can change also the speed of convergence (of the implemented simulation algorithm). In particular, decreasing $\bar{p}_{\text{sense}}$ and $\bar{p}_{\text{move}}$ leads to a faster convergence, whereas increasing them leads to a slower convergence. Here we presented simulation results of the model in (1). Simulation of different swarm models can be found in [18] and [19] (and the references therein).

VI. Concluding Remarks

In this correspondence, we present a 1-D asynchronous swarm model and analyze its stability. The model can also include sensing or communication delays and *total asynchronism*. We show that for our model we have asymptotic convergence of the positions of the swarm members to the comfortable position despite the presence of delays and asynchronism. In our analysis we use tools from the parallel and distributed computation literature. These tools are important since it might be possible to use them for analysis of n-dimensional swarms as well as n-dimensional synchronization problems. Future research can focus on these directions as well as on moving swarms.

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