The Linear Output Regulation Problem

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Acknowledgments

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- A large part of this series of lectures is the result of a several years of joint work with Alberto Isidori (Washington University, St. Louis and Università di Roma) and Lorenzo Marconi (Università di Bologna). Their contribution has been invaluable, and hopefully their influence would be visible throughout the course.

Outline of the Course

- Thursday, May 26
  - The Linear Output Regulation Problem
  - Nonlinear Local and Structurally Stable Regulation
- Friday, May 27
  - Robust and Adaptive Nonlinear Regulation in the Large
  - Application: Helicopter Landing

Linear Regulation - Outline

- Problem Formulation

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- Solution to the Full-Information Problem

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- Solution to the Error-Feedback Problem
- The Internal Model Principle
- The Construction of a Robust Regulator

Problem formulation

Consider a linear plant model of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) + Pw(t) \\ e(t) &= Cx(t) + Qw(t)\end{aligned}$$

with state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$, and error to be regulated $e \in \mathbb{R}^m$.

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The signal $w \in \mathbb{R}^d$ is generated by a linear exosystem

$$\dot{w}(t) = Sw(t)$$

The exogenous signal $w(t)$ includes references to be tracked and disturbances to be rejected.

Problem Formulation

The problem is to find a control law such that:

- The origin is an asymptotically stable equilibrium when the exosystem is disconnected, that is when $w(t) \equiv 0$;

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- The error $e(t)$ converges to zero, for any initial condition of the plant and the exosystem.

Note that the first requirement allows to restrict the analysis to the case in which

$$\operatorname{spec}\{S\} \subset \overline{\mathbb{C}^+}.$$

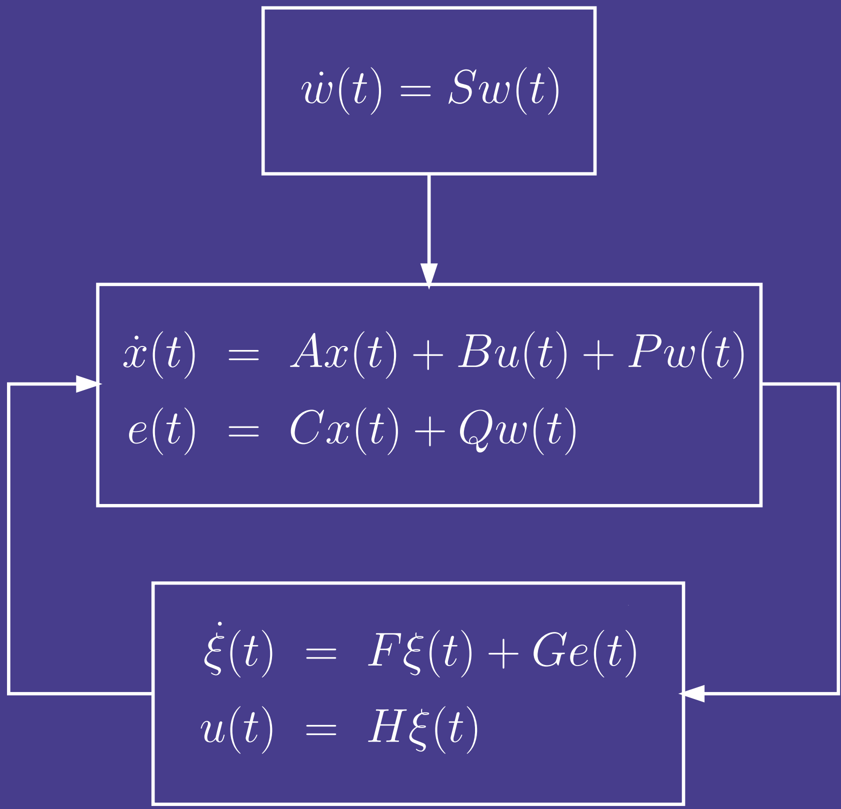
Problem Formulation

Typically, the error $e(t)$ is the only variable assumed to be available for measurement *(error-feedback regulation.)* In this case, we look for a dynamic controller of the form

$$\begin{aligned}\dot{\xi}(t) &= F\xi(t) + Ge(t)\\ u(t) &= H\xi(t)\end{aligned}$$

with state $\xi \in \mathbb{R}^\nu$.

For the time being, we assume that the plant and the exosystem models are known accurately.



Error-Feedback Regulation Problem

The EF regulation problem is stated as follows:

Given $\{A, B, C, P, Q, S\}$, find $\{F, G, H\}$ such that

- The closed-loop matrix $\begin{pmatrix} A & BH \\ GC & F \end{pmatrix}$ is Hurwitz.

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- For any initial condition, the trajectory of

$$\begin{aligned} \dot{x} &= Ax + BH\xi + Pw \\ \dot{\xi} &= GCx + F\xi + GQw \\ \dot{w} &= Sw \end{aligned}$$

satisfies $\lim_{t \to \infty} (Cx(t) + Qw(t)) = 0.$

Full-Information Regulation Problem

It is convenient to state the *full-information problem* , where it is assumed that both x and w are available for feedback, and the control is $u(t) = Kx(t) + Lw(t)$.

Given $\{A, B, C, P, Q, S\}$, find $\{K, L\}$ such that

- The matrix $A + BK$ is Hurwitz.

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Given $\{A, B, C, P, Q, S\}$, find $\{K, L\}$ such that

- The matrix $A + BK$ is Hurwitz.
- For any initial condition, the trajectory of

$$\begin{aligned} \dot{x} &= (A + BK)x + (BL + P)w \\ \dot{w} &= Sw \end{aligned}$$

satisfies $$\lim_{t \to \infty} (Cx(t) + Qw(t)) = 0.$$

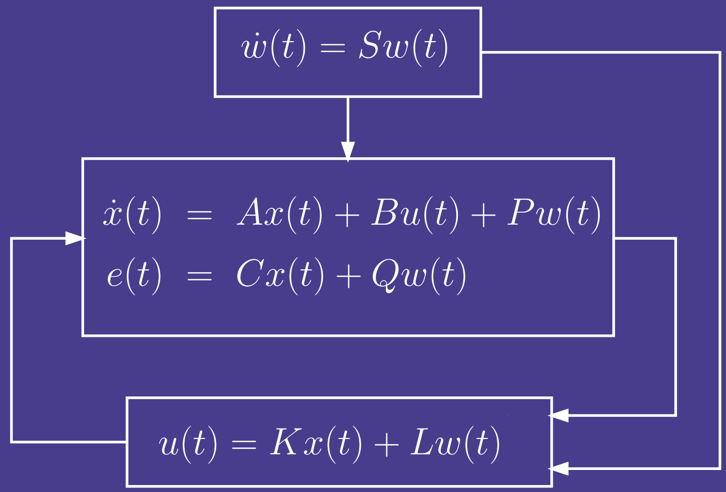
Solution of the Full-Information Problem

Assume that (A, B) is stabilizable, and apply a control of the form

$$u(t) = Kx(t) + Lw(t)$$

where K is such that $A + BK$ is Hurwitz.

- How should one choose L so that the given control law provides regulation?



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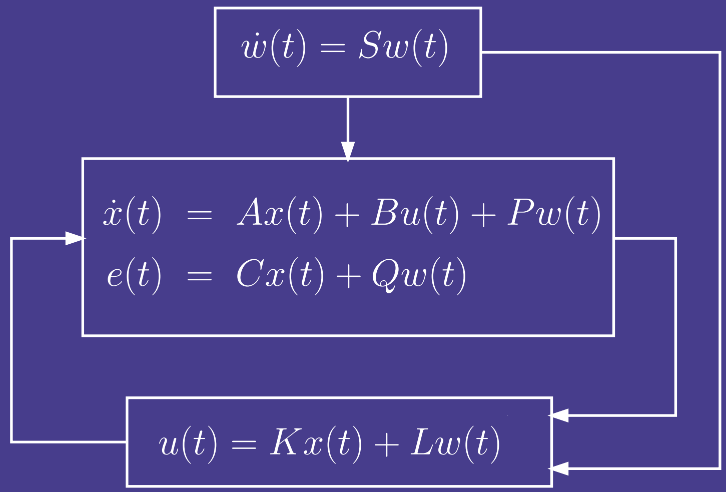

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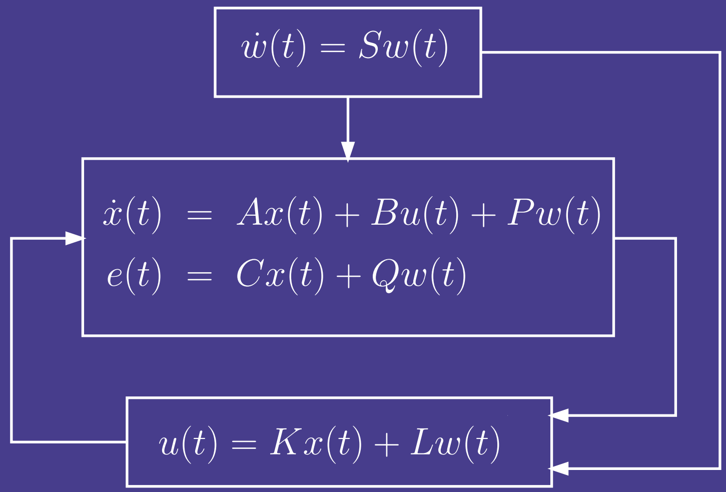
- How should one choose L so that the given control law provides regulation?
- Can one find such L at all?

This depends on $\{A, B, C, P, Q, S\}$.

Note that K has already been fixed.



Solution of the Full-Information Problem

By assumption, the closed-loop matrix

$$A_{cl} = \begin{pmatrix} A + BK & P + BL \\ 0 & S \end{pmatrix}$$

has n eigenvalues in $\mathbb{C}^-$ and d eigenvalues in $\overline{\mathbb{C}^+}$,

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$$\mathcal{V}^- = \mathrm{Im} \begin{pmatrix} I_n \\ 0 \end{pmatrix}, \qquad \mathcal{V}^+ = \mathrm{Im} \begin{pmatrix} \Pi \\ I_d \end{pmatrix}$$

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for some $\Pi \in \mathbb{R}^{n \times d}$. Note that

$$A_{cl}|_{\mathcal{V}^-} = A + BK, \qquad A_{cl}|_{\mathcal{V}^+} = S.$$

Solution of the Full-Information Problem

Since $\mathcal{V}^+$ is A_{cl}-invariant,

$$A_{cl}\mathcal{V}^+ \subseteq \mathcal{V}^+ \iff \forall\, w \in \mathbb{R}^d \;\; \exists\, \tilde{w} \in \mathbb{R}^d \text{ such that}$$

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$$(A + BK)\Pi + P + BL = \Pi S.$$

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Sylvester equation: the solution Π is unique, since the spectra of $A + BK$ and S are disjoint.

Solution of the Full-Information Problem

The subspace

$$\mathcal{V}^{+} = \left\{(x, w) \in \mathbb{R}^{n+d} : x = \Pi w\right\}$$

defines a globally attractive steady-state of the closed-loop system.

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$$\tilde{x} = x - \Pi w$$

we write

$$\begin{aligned} \dot{\tilde{x}} &= (A + BK)\tilde{x} \\ \dot{w} &= Sw \\ e &= C\tilde{x} + (C\Pi + Q)w \end{aligned}$$

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and

$$e(t) = C\mathrm{e}^{(A+BK)t}\tilde{x}_0 + (C\Pi + Q)\mathrm{e}^{St}w_0.$$

Solution of the Full-Information Problem

Since

$$\lim_{t\to\infty} e(t) = \lim_{t\to\infty} (C\Pi + Q)\mathrm{e}^{St} w_0$$

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- The term Kx is selected to <u>create</u> a globally attractive steady state
- The term Lw must be selected to <u>shape</u> the steady state

The Regulator Equations

Theorem 1 (Francis, 1977) *Let* (A, B) *be stabilizable. The FI problem is solvable if and only if there exist* $\Pi \in \mathbb{R}^{n\times d}$ *and* $R \in \mathbb{R}^{m\times d}$ *solution of the regulator equations*

$$\begin{aligned} \Pi S &= A\Pi + BR + P \\ 0 &= C\Pi + Q\,, \end{aligned}$$

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$$\begin{aligned} \Pi S &= A\Pi + BR + P \\ 0 &= C\Pi + Q\,, \end{aligned}$$

or, equivalently, if and only if the system

$$\begin{aligned} \dot{x} &= Ax + Pw + Bu \\ \dot{w} &= Sw \\ e &= Cx + Qw \end{aligned}$$

admits a controlled-invariant subspace $\mathcal{V} \subset \operatorname{Ker}\begin{pmatrix} C & Q \end{pmatrix}$.

Feedback and Feedforward

Note that the existence of the solution is independent of K, which has the only role of stabilizing the system. Fix K, and choose

$$L = R - K\Pi$$

to obtain

$$u = K(x - \Pi w) + Rw = K\tilde{x} + Rw$$

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Feedback control: steers $x(t)$ to $\mathcal{V}$

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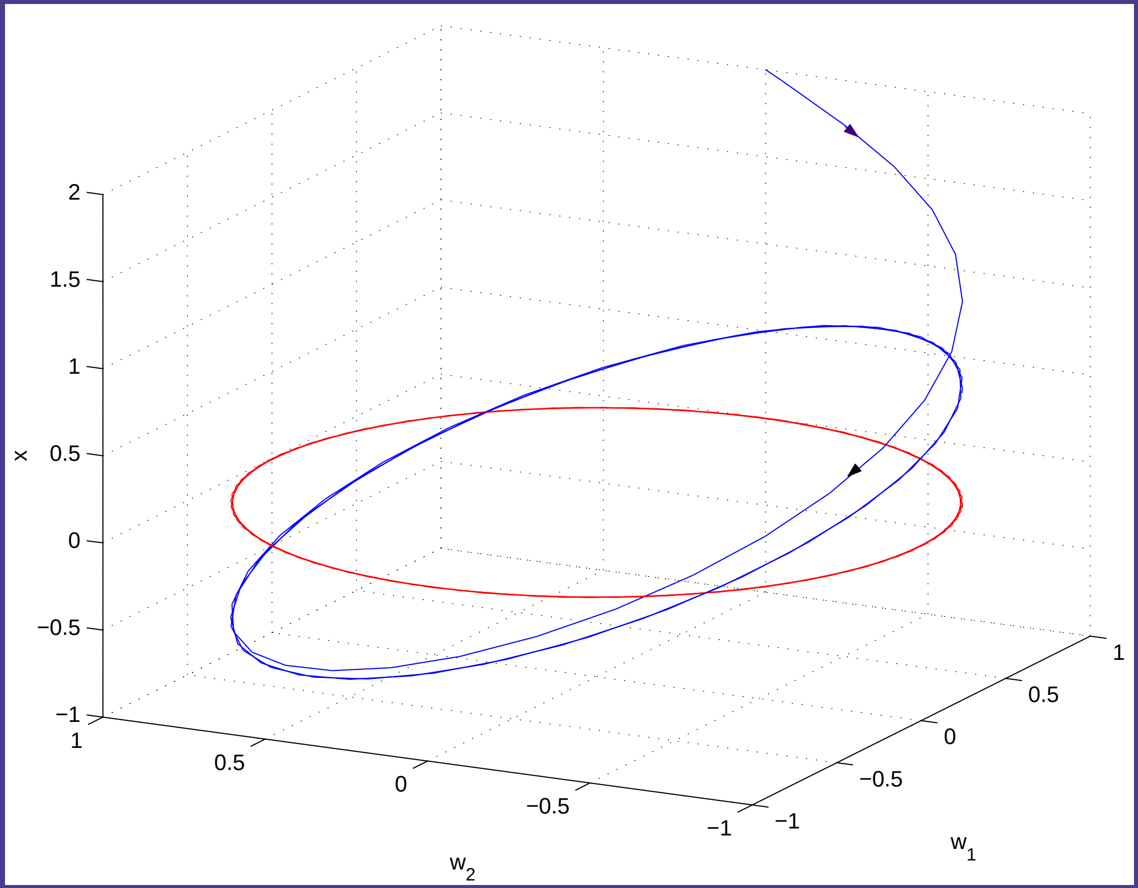
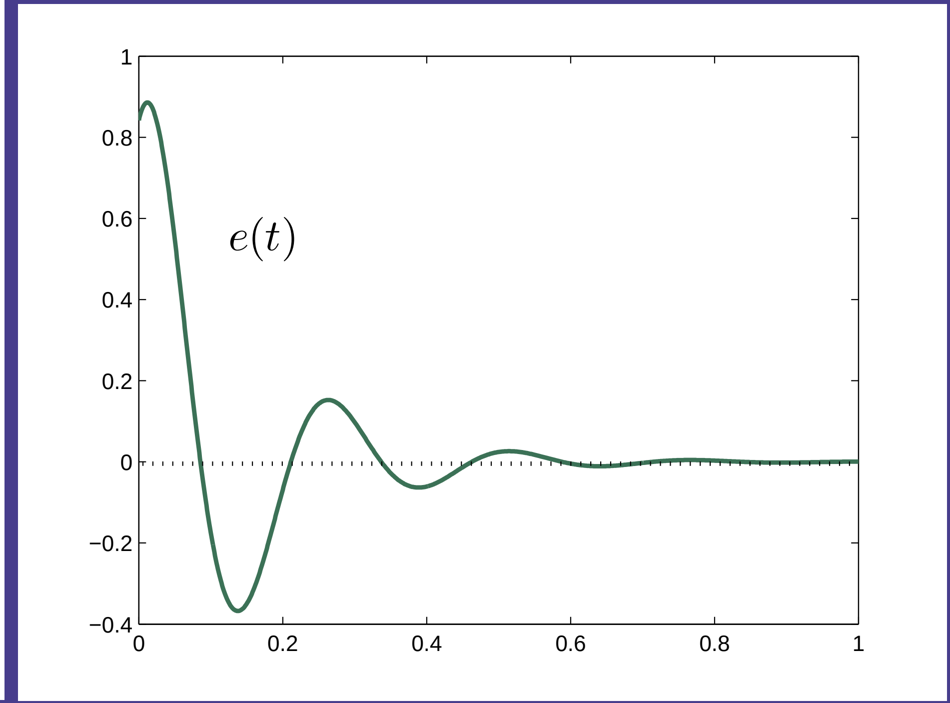
Feedforward control: renders $\mathcal{V}$ invariant.

The Geometric Picture

- Rendering an appropriate subspace $\mathcal{V} \subset \mathrm{Ker}\begin{pmatrix} C & Q \end{pmatrix}$ invariant and attractive $\iff \lim_{t\to\infty} e(t) = 0$





The Error-Feedback Case

Consider now the case of an error-feedback controller

$$\begin{aligned} \dot{\xi}(t) &= F\xi(t) + Ge(t) \\ u(t) &= H\xi(t) \end{aligned}$$

such that $A_a = \begin{pmatrix} A & BH \\ GC & F \end{pmatrix}$ is Hurwitz.

- Under which conditions does a stabilizing controller controller provide asymptotic regulation?

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- Closed-loop matrix:

$$
A_{cl} = \begin{pmatrix} A & BH & P \\ GC & F & GQ \\ 0 & 0 & S \end{pmatrix}
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$$A_{cl} = \left(\begin{array}{cc|c} A & BH & P \\ GC & F & GQ \\ \hline 0 & 0 & S \end{array}\right) \quad \mathcal{V}^- = \mathrm{Im}\begin{pmatrix} I_n \\ I_\nu \\ 0 \end{pmatrix}, \mathcal{V}^+ = \mathrm{Im}\begin{pmatrix} \Pi \\ \Sigma \\ I_d \end{pmatrix}$$

The Error-Feedback Case

The corresponding Sylvester equation

$$\begin{pmatrix} A & BH \\ GC & F \end{pmatrix} \begin{pmatrix} \Pi \\ \Sigma \end{pmatrix} + \begin{pmatrix} P \\ GQ \end{pmatrix} = \begin{pmatrix} \Pi \\ \Sigma \end{pmatrix} S$$

has a unique solution $(\Pi,\ \Sigma)$.

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$$\begin{aligned} A\Pi + BH\Sigma + P &= \Pi S \\ GC\Pi + F\Sigma + GQ &= \Sigma S. \end{aligned}$$

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$$F\Sigma = \Sigma S.$$

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$$\begin{aligned} F\Sigma &= \Sigma S \\ H\Sigma &= R \end{aligned}$$

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The first 2 equations are exactly the regulator equations.

The Internal Model Principle

Theorem 2 (Francis, 1977) *Let (A, B) be stabilizable and (A, C) detectable. The given controller solves the EF problem if and only if there exist $\Pi \in \mathbb{R}^{n\times d}$, $R \in \mathbb{R}^{m\times d}$ and $\Sigma \in \mathbb{R}^{\nu\times d}$ such that*

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$$
\begin{aligned}
\Sigma S &= F\Sigma \\
R &= H\Sigma
\end{aligned}
$$

The last 2 equations constitute the *internal model principle*.

The controller must generate internally the feedforward control required to render the error-zeroing subspace invariant.

Regulator Synthesis

Indeed, the solvability of the FI problem alone is necessary and sufficient for the solution of the EF problem. Assume that:

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(c) There exist Π and R solution of the regulator equations.

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Note. It can be shown that assuming (b) in place of detectability of (A, C) does not involve any loss of generality.

Regulator Synthesis

Assumption (b) implies the existence of an observer for $(x,\ w)$.

The FI control

$$u_{FI} = Kx + (R - K\Pi)w$$

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The FI control

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is replaced by the certainty equivalence controller

$$\begin{pmatrix} \dot{\xi}_0 \\ \dot{\xi}_1 \end{pmatrix} = \left[\begin{pmatrix} A & P \\ 0 & S \end{pmatrix} - \begin{pmatrix} G_0 \\ G_1 \end{pmatrix} \begin{pmatrix} C & Q \end{pmatrix} \right] \begin{pmatrix} \xi_0 \\ \xi_1 \end{pmatrix} + \begin{pmatrix} G_0 \\ G_1 \end{pmatrix} e + \begin{pmatrix} B \\ 0 \end{pmatrix} u$$

$$u_{EF} = K\xi_0 + (R - K\Pi)\xi_1,$$

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$$u_{EF} = K\xi_0 + (R - K\Pi)\xi_1,$$

where G_0 and G_1 are such that

$$\operatorname{spec}\left\{ \begin{pmatrix} A & P \\ 0 & S \end{pmatrix} - \begin{pmatrix} G_0 \\ G_1 \end{pmatrix} (C\ Q) \right\} \subset \mathbb{C}^-$$

Regulator Synthesis

The regulator $\{F, G, H\}$ is given by

$$F = \begin{pmatrix} A - G_0C + BK & P - G_0Q + B(R - B\Pi) \\ -G_1C & S - G_1Q \end{pmatrix}$$

$$G = \begin{pmatrix} G_0 \\ G_1 \end{pmatrix}, \qquad H = \begin{pmatrix} K & R - K\Pi \end{pmatrix}$$

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- <u>Stabilization:</u>

$$A_{cl} \quad \text{similar to} \quad \begin{pmatrix} A + BK & BK & B(R - K\Pi) \\ 0 & A - G_0 C & P - G_0 Q \\ 0 & -G_1 C & S - G_1 Q \end{pmatrix}$$

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- Regulation:

$$\begin{aligned} A\Pi + BR + P &= \Pi S \\ C\Pi + Q &= 0 \end{aligned}$$

hold by assumption.

Regulator Synthesis

The regulator $\{F, G, H\}$ is given by

$$F = \begin{pmatrix} A - G_0C + BK & P - G_0Q + B(R - B\Pi) \\ -G_1C & S - G_1Q \end{pmatrix}$$

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- Regulation: Letting $\Sigma = \begin{pmatrix} \Pi^{\mathrm{T}} & I_d \end{pmatrix}^{\mathrm{T}}$ we obtain

$$\begin{aligned} F\Sigma &= \begin{pmatrix} A - G_0C + BK & P - G_0Q + B(R - B\Pi) \\ -G_1C & S - G_1Q \end{pmatrix} \begin{pmatrix} \Pi \\ I_d \end{pmatrix} \\ &= \Sigma S. \end{aligned}$$

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- Regulation: Letting $\Sigma = \begin{pmatrix} \Pi^{\mathrm{T}} & I_d \end{pmatrix}^{\mathrm{T}}$ we obtain

$$\begin{aligned} H\Sigma &= K\Pi + R - K\Pi \\ &= R \end{aligned}$$

The Issue of Robustness

Consider an uncertain plant model of the form

$$\begin{aligned}\dot{x} &= A(\mu)x + B(\mu)u + P(\mu)w \\ e &= C(\mu)x + Q(\mu)w\end{aligned}$$

where $\mu \in \mathcal{P} \subset \mathbb{R}^p$, and $\mathcal{P}$ is compact.

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The problem is to find a fixed controller $\{F, G, H\}$ such that

- The matrix $\begin{pmatrix} A(\mu) & B(\mu)H \\ GC(\mu) & F \end{pmatrix}$ is Hurwitz for all $\mu \in \mathcal{P}$.
- For any initial condition and for all $\mu \in \mathcal{P}$, the trajectory of the closed-loop system satisfies

$$\lim_{t\to\infty} \left(C(\mu)x(t) + Q(\mu)w(t)\right) = 0.$$

The Issue of Robustness

A necessary condition is that the plant is robustly stabilizable.

- Is it possible to design a robustly stabilizing controller that achieves regulation $\forall\, \mu \in \mathcal{P}$?

The question is not trivial: (exponential) stabilization is an "open" property, regulation need not be.

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Recall the two ingredients of output regulation:

- Feedback control (intrinsically robust: does not require accurate knowledge of the parameters)
- Feedforward control (intrinsically fragile: depends on the actual value of the plant parameters)

The Robust Regulator

Theorem 3 *Assume $\{F, G, H\}$ is a robust stabilizer. The given controller is a robust regulator if and only if there exist $\Pi(\mu) \in \mathbb{R}^{n\times d}$, $R(\mu) \in \mathbb{R}^{m\times d}$ and $\Sigma(\mu) \in \mathbb{R}^{\nu\times d}$ such that*

$$(FI) \qquad \left\{ \begin{array}{rcl} \Pi(\mu)S & = & A(\mu)\Pi + B(\mu)R(\mu) + P(\mu) \\ 0 & = & C(\mu)\Pi(\mu) + Q(\mu) \end{array} \right.$$

$$(IM) \qquad \left\{ \begin{array}{rcl} \Sigma(\mu)S & = & F\Sigma(\mu) \\ R(\mu) & = & H\Sigma(\mu) \end{array} \right.$$

hold for all $\mu \in \mathcal{P}$.

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hold for all $\mu \in \mathcal{P}$.

- We need conditions that guarantee (FI)
- We need to design an internal model that enforces (IM)

The Robust Regulator

The (FI) condition is satisfied if

$$\det \begin{pmatrix} A(\mu) - \lambda I & B(\mu) \\ C(\mu) & 0 \end{pmatrix} \neq 0$$

for all $\lambda \in \mathrm{spec}\{S\}$ and all $\mu \in \mathcal{P}$.

This amounts in requiring that the set of transmission zeros of the plant is disjoint from the set of eigenvalues of the exosystem (non-resonance condition).

Henceforth, we assume that this is the case.

The Construction of a Robust Regulator

Consider the minimal polynomial of S

$$m(\lambda) = \lambda^q + a_{q-1}\lambda^{q-1} + \cdots + a_1\lambda + a_0.$$

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and thus, for any $\mu \in \mathcal{P}$

$$R(\mu)S^q = -(a_1 R(\mu)S^{q-1} + \cdots + a_1 R(\mu)S + a_0 R(\mu))\,.$$

The Construction of a Robust Regulator

Define the pair (Φ, Γ) as

$$\Phi = \begin{pmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ -a_0 I & -a_1 I & \cdots & -a_{q-1} I \end{pmatrix}, \quad \Gamma = \begin{pmatrix} I & 0 & \cdots & 0 \end{pmatrix}$$

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and note that the matrix $T(R) \in \mathbb{R}^{mq \times d}$ defined as

$$T(R) = \begin{pmatrix} R \\ RS \\ \cdots \\ RS^{q-1} \end{pmatrix}$$

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$$T(R) = \begin{pmatrix} R \\ RS \\ \cdots \\ RS^{q-1} \end{pmatrix}$$ satisfies $$\boxed{\begin{aligned} T(R(\mu))S &= \Phi T(R(\mu)) \\ R(\mu) &= \Gamma T(R(\mu)) \end{aligned}}$$

for all $\mu \in \mathcal{P}$.

The Construction of a Robust Regulator

The $(d+p)$-dim system

$$\begin{aligned} \dot{\mu} &= 0 \\ \dot{w} &= Sw \\ u_{ff} &= R(\mu)w \end{aligned} \tag{1}$$

generating the (unknown) feedforward input

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in the sense that every output trajectory of (1) is an output trajectory of (2):

$$\eta(0) = T(R(\mu))w(0) \Longrightarrow \Gamma\eta(t) = R(\mu)w(t) \quad \forall\, t \geq 0$$

The Construction of a Robust Regulator

The candidate controller is selected as

$$F = \begin{pmatrix} \Phi & 0 \\ 0 & L \end{pmatrix}, \qquad G = \begin{pmatrix} \Theta \\ M \end{pmatrix}, \qquad H = \begin{pmatrix} \Gamma & N \end{pmatrix}$$

- (Φ, Γ) have been already defined

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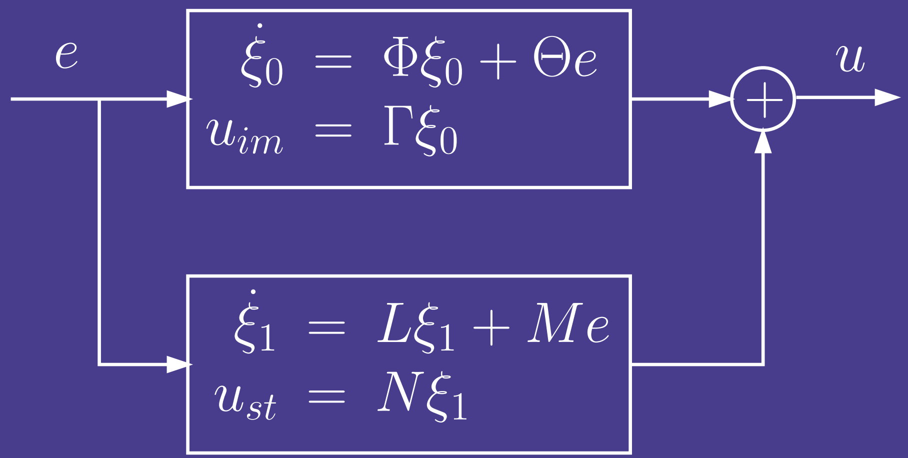
$$\begin{aligned} \dot{\xi}_0 &= \Phi\xi_0 + \Theta e \\ \dot{\xi}_1 &= L\xi_1 + Me \\ u &= \Gamma\xi_0 + N\xi_1 \end{aligned}$$



The Error System

Look at the closed-loop system

$$\begin{aligned}\dot{w} &= Sw \\ \dot{\xi}_0 &= \Phi\xi_0 + \Theta e \\ \dot{\xi}_1 &= L\xi_1 + Me \\ \dot{x} &= A(\mu)x + P(\mu)w + B(\mu)[\Gamma\xi_0 + N\xi_1] \\ e &= C(\mu)x + Q(\mu)w\end{aligned}$$

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$$\begin{aligned} \dot{w} &= Sw \\ \dot{\xi}_0 &= \Phi\xi_0 + \Theta C(\mu)\tilde{x} \\ \dot{\xi}_1 &= L\xi_1 + MC(\mu)\tilde{x} \\ \dot{\tilde{x}} &= A(\mu)\tilde{x} + B(\mu)[\Gamma\xi_0 - R(\mu)w + N\xi_1] \\ e &= C(\mu)\tilde{x} \end{aligned}$$

change coordinates as

$$\tilde{x} = x - \Pi(\mu)w$$

The Error System

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change coordinates as

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and rearrange the equations to obtain the error system

The Error System

Look at the closed-loop system as the interconnection of the error system and a robust stabilizer

$$
\begin{aligned}
\dot{\tilde{\xi}}_0 &= \Phi\tilde{\xi}_0 + \Theta C(\mu)\tilde{x} \\
\dot{\tilde{x}} &= A(\mu)\tilde{x} + B(\mu)\Gamma\tilde{\xi}_0 + B(\mu)v \\
e &= C(\mu)\tilde{x}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\xi}_1 &= L\xi_1 + Me \\
v &= N\xi_1
\end{aligned}
$$

robust stabilization implies robust regulation.

The Error System

Lemma 1 *Suppose the pair* $(A(\mu), B(\mu))$ *is stabilizable and the pair* $(C(\mu), A(\mu))$ *is detectable. Suppose*

$$\det \begin{pmatrix} A(\mu) - \lambda I & B(\mu) \\ C(\mu) & 0 \end{pmatrix} \neq 0\,, \qquad \forall\, \lambda \in \mathrm{spec}\{S\}\,, \quad \forall\, \mu \in \mathcal{P}\,.$$

Let Φ *and* Γ *be as defined, and* Θ *such that the pair* (Φ, Θ) *is controllable. Then, the triplet*

$$\begin{pmatrix} \Phi & \Theta C(\mu) \\ B(\mu)\Gamma & A(\mu) \end{pmatrix}, \qquad \begin{pmatrix} 0 \\ B(\mu) \end{pmatrix}, \qquad \begin{pmatrix} 0 & C(\mu) \end{pmatrix}$$

is stabilizable and detectable for all $\mu \in \mathcal{P}$.

Conclusions

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- The internal model is not just an observer of the exosystem.
- Robust regulation is a byproduct of robust stabilization.
- What can we carry over to nonlinear systems?