A Taxonomy for Time-Varying Immersions in Periodic Internal-Model Control

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Outline of the Talk

1. Statement of the Problem
   - Regulation in the Periodic Setup
2. Solvability of the Problem
   - Periodic Internal Model Principle
   - Periodic Immersions
3. Robust Internal Model Design
   - Canonical Realizations
   - Regulator Structure
4. Adaptive Internal Model Design
   - Canonical Parameterization
   - Adaptive Regulator Structure
   - Weak Immersions for Adaptive Design
5. Illustrative Example
6. Conclusions

Milestones in Output Regulation Theory

1. Davison, E. J.; Goldenberg, A., “Robust control of a general servomechanism problem: The servo compensator,” *Automatica*, 1975.
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Regulation in the Periodic Setup

- Regulation of parametrized families of linear T-periodic systems:

$$\begin{aligned} \dot{w} &= S(t,\sigma)w \\ \dot{x} &= A(t,\mu)x + B(t,\mu)u + P(t,\mu)w \\ e &= C(t,\mu)x + Q(t,\mu)w\,, \end{aligned} \qquad (1)$$

  - exosystem state $w \in \mathbb{R}^{n_w}$, plant state $x \in \mathbb{R}^n$
  - control input $u \in \mathbb{R}$, and regulated error $e \in \mathbb{R}$
  - parameter vectors $(\sigma,\mu) \in \mathcal{K}_\sigma \times \mathcal{K}_\mu \subset \mathbb{R}^s \times \mathbb{R}^p$

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- Look for a parameterized family of T-periodic controllers

$$
\begin{aligned}
\dot{\xi} &= F(t,\theta)\xi + G(t,\theta)e \\
u &= H(t,\theta)\xi + K(t,\theta)e\,,
\end{aligned} \tag{2}
$$

with state $\xi \in \mathbb{R}^\nu$ and tunable parameter vector $\theta \in \mathcal{K}_\theta \subset \mathbb{R}^\rho$.

Certainty Equivalence Control

The controller (2) is a ***certainty equivalence controller*** if $\forall\, \mu \in \mathcal{K}_\mu$:

1. The unforced closed-loop system

$$\begin{aligned} \dot{x} &= \big[A(t,\mu) + B(t,\mu)K(t,\theta)C(t,\mu)\big]x + B(t,\mu)H(t,\theta)\xi \\ \dot{\xi} &= F(t,\theta)\xi + G(t,\theta)C(t,\mu)x \end{aligned}$$

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2. There exists a continuous assignment $\sigma \mapsto \theta_\sigma$ such that for any given $\sigma \in \mathcal{K}_\sigma$, the fixed controller

$$\begin{aligned} \dot{\xi} &= F(t,\theta_\sigma)\xi + G(t,\theta_\sigma)e \\ u &= H(t,\theta_\sigma)\xi + K(t,\theta_\sigma)e \end{aligned}$$

solves the robust output regulation problem for (1), i.e., boundedness of all trajectories, and $\lim_{t\to\infty} e(t) = 0$. □

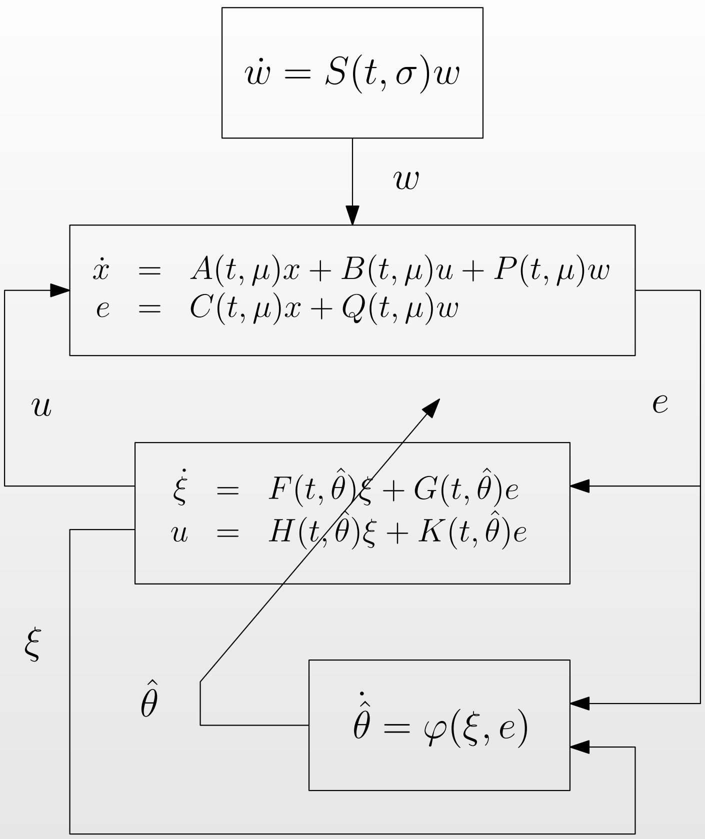
Adaptive Regulation

Once a certainty-equivalence has been found, look for an update law

$$\dot{\hat{\theta}} = \varphi(\xi, e)$$

to tune the family of controllers to the one achieving regulation.

Note that adaptation is not used for stabilization.



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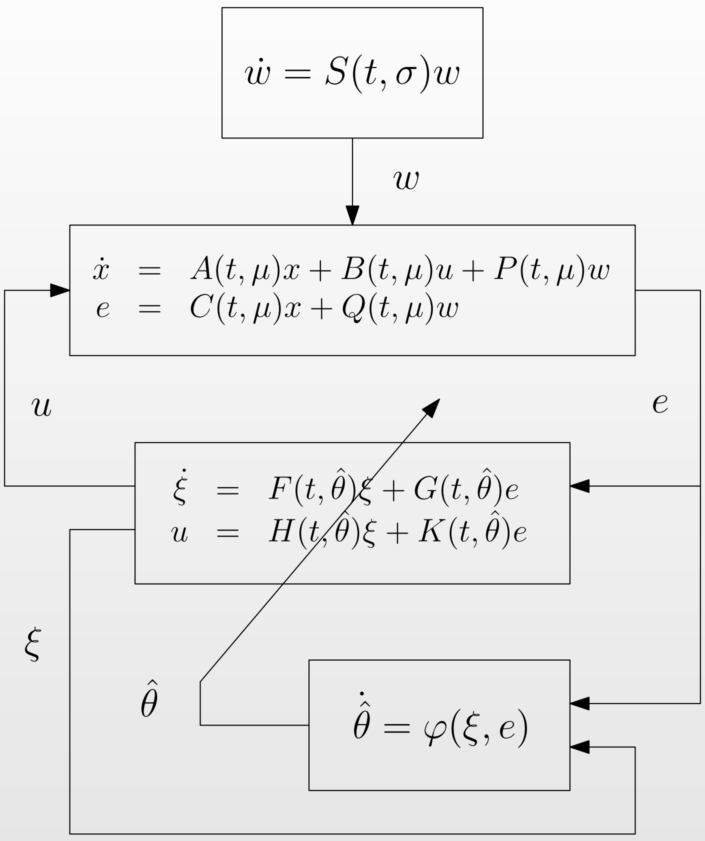

Issues

- Find a characterization of all certainty-equivalence regulators (**canonical realization**)

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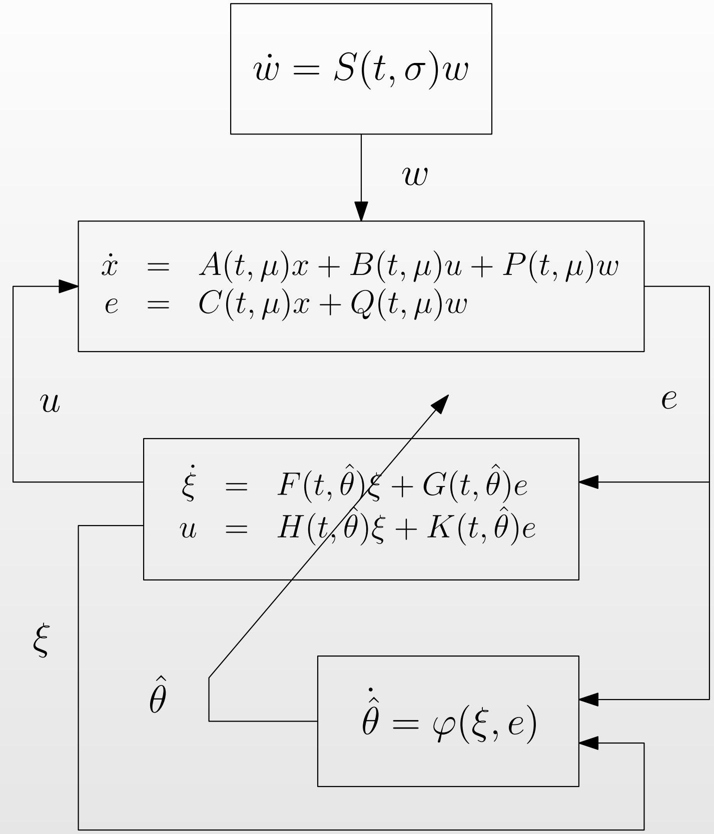

Issues

- Find a characterization of all certainty-equivalence regulators (**canonical realization**)
- Find a parameterization that is amenable to adaptive control (**canonical parameterization**)

Periodic Internal Model Principle

Assume σ fixed (look at robust regulation first).

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for all $t \in [0, T)$ and all $\mu \in \mathcal{K}_\mu$.

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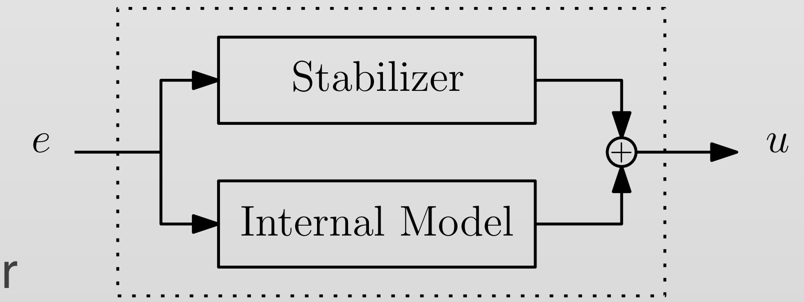
The periodic feed-forward control

$$\begin{aligned} \dot{w} &= S(t)w \\ v &= R(t,\mu)w \end{aligned}$$

must be embedded in the controller



Periodic Immersion

The parameterized family of periodic systems $(S(t), R(t,\mu))$ is **immersed** into $(\Phi(t), \Gamma(t))$ if there exists a periodic map Υ such that:

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1. and 2. are not equivalent (2.⇒1.) even for periodic systems
2. ⇒ existence of **observer and observability canonical forms**
3. is useful only for adaptive regulation

Examples

The periodic exosystem

$$S(t) = \begin{pmatrix} 0 & \sin(t) \\ -\sin(t) & 0 \end{pmatrix}, \quad R(\mu) = \begin{pmatrix} \mu_1 & \mu_2 \end{pmatrix}, \quad \|\mu\|^2 = 1$$

- Is UCO, but not UO, since $\det \mathcal{O} = \sin(t)$
- Is immersed into a 3-dim UO system in observability form

$$\Phi(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3\sin(t)\cos(t) & -1-\sin^2(t) & 0 \end{pmatrix}, \quad \Gamma = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$$

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Assuming that a **regular immersion** exists, how does one construct and *internal model*?

Canonical Realization of the Internal Model

The **regular** internal-model pair $(\Phi(\cdot), \Gamma(\cdot))$ is said to admit a *canonical realization* if there exist a periodic map $M(\cdot)$ and a periodic system $(F_{\rm im}(\cdot), G_{\rm im}(\cdot), H_{\rm im}(\cdot))$ such that:

1. $F_{\rm im}(t) \in \mathbb{R}^{m\times m}$ has all characteristic multipliers in $|\lambda| < 1$
2. $M(t)$ has constant rank, and satisfies for all $t \in [0, T)$

$$\begin{aligned} \dot M(t) + M(t)\Phi(t) &= (F_{\rm im}(t) + G_{\rm im}(t)H_{\rm im}(t))M(t) \\ \Gamma(t) &= H_{\rm im}(t)M(t) \end{aligned}$$

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Internal model unit

$$\begin{aligned} \dot\xi &= F_{\rm im}(t)\xi + G_{\rm im}(t)u \\ u &= H_{\rm im}(t)\xi + u_{\rm st} \longleftarrow \textit{stabilizer} \end{aligned}$$

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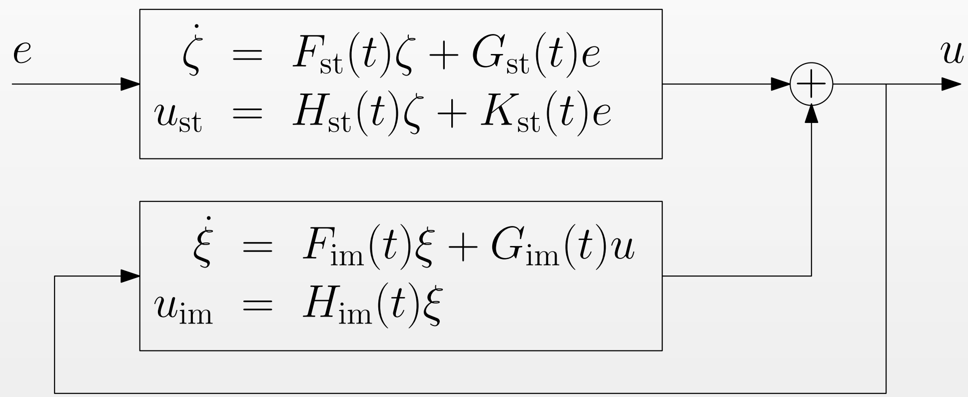
In a nutshell, the canonical realization yields for the IMU:

"zeros" in $|\lambda| = 1$ (characteristic multipliers of $F_{\rm im} + G_{\rm im}H_{\rm im}$)
"poles" in $|\lambda| < 1$ (characteristic multipliers of $F_{\rm im}$).

Structure of the Robust Regulator



Regular IM pair $(\Phi(\cdot), \Gamma(\cdot))$	*Strong IM pair* $(\Phi_o(\cdot), \Gamma_o)$
$F_{im}(t) = -\alpha I - \Phi'(t),\ \alpha > 0$	$F_{im}(t) = F_{im}$ Hurwitz
$G_{im}(t) = \Gamma'(t)$	$G_{im}(t) = Q^{-1}[\phi_1(t) + b]$
$H_{im}(t) = \Gamma(t)M^{-1}(t)$	$H_{im}(t) = H_{im}$ observable pair

- For regular IM pairs, it is a passivity-based design
  - closed-loop eigenvalues not free
  - requires the computation of $M^{-1}(t)$
- For strong IM pairs, eigenvalues are assigned via output injection

Canonical Parameterization of the Internal Model

The family of internal-model pairs $\big(\Phi(\cdot,\sigma),\Gamma(\cdot,\sigma)\big)$ is said to admit a *canonical parametrization in feedback form* if there exist a family of periodic maps $M(\cdot,\theta)$ and a family of periodic systems $(F_{\rm im}(\cdot),G_{\rm im}(\cdot),H_{\rm im}(\cdot,\theta))$ such that:

1. $F_{\rm im}(t)$ has all characteristic multipliers in $|\lambda|<1$
2. $H_{\rm im}(t,\theta)$ is affine in θ
3. there exists a continuous map $\sigma\mapsto\theta_\sigma$ such that the matrix $M(t,\theta_\sigma)$ has constant rank $\forall\, t\in[0,T)$ and $\forall\,\sigma\in\mathcal{K}_\sigma$, and

$$
\begin{aligned}
\dot M(t,\theta_\sigma)+M(t,\theta_\sigma)\Phi(t,\sigma) &= (F_{\rm im}(t)+G_{\rm im}(t)H_{\rm im}(t,\theta_\sigma))M(t,\theta_\sigma)\\
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Note that:

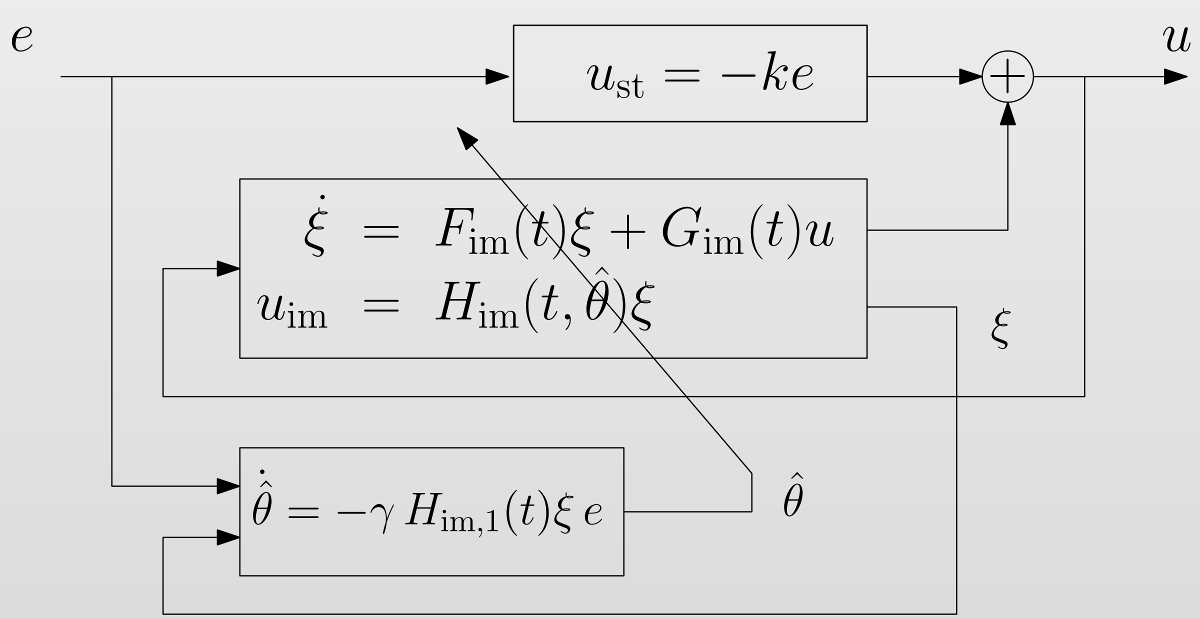
- $F_{\rm im}(t)$ and $G_{\rm im}(t)$ are both independent of θ
- $H_{\rm im}(t,\theta) = H_{\rm im,\,0}(t) + \theta^{\rm T}\, H_{\rm im,\,1}(t)$

Structure of the Adaptive Regulator

Adaptive internal model unit (for relative-degree 1 minimum-phase plants)

$$\begin{aligned} \dot\xi &= F_{\rm im}(t)\xi + G_{\rm im}(t)u \\ \dot{\hat\theta} &= -\gamma\, H_{{\rm im},1}(t)\xi\, e, \quad \gamma > 0 \\ u &= H_{\rm im}(t,\hat\theta)\xi + u_{\rm st} \end{aligned}$$

$\longleftarrow$ *stabilizer*



How to Construct a Canonical Parameterization?

- Start from a strong immersion $\Rightarrow (\Phi_o(\cdot,\sigma), \Gamma_o)$ in observer form
- Find a linear parameterization of the first column of $\Phi_o(\cdot,\sigma)$

$$\Phi_{\rm o}(t,\sigma) = \Phi_{\rm b} - \Theta\beta(t)\Gamma_{\rm o}$$

$$\Phi_{\rm b} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \Theta = \begin{pmatrix} \theta_{q-1}^{\rm T} \\ \vdots \\ \theta_1^{\rm T} \\ \theta_0^{\rm T} \end{pmatrix}.$$

- Choose L_0 such that $F = \Phi_{\rm b} - L_0\Gamma_{\rm o}$ is Hurwitz
- Let $G(t,\theta) = L_0 - \Theta\beta(t)$, $H = \Gamma_{\rm o}$
  - The triplet $(F, G(\cdot,\theta), H)$ is not yet in *feedback form*.
  - We need to “shift” the parameter θ from G to H
  - Impossible to do with a change of coordinates!

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- Start from a strong immersion $\Rightarrow (\Phi_o(\cdot,\sigma), \Gamma_o)$ in observer form
- Find a linear parameterization of the first column of $\Phi_o(\cdot,\sigma)$

$$\Phi_\mathrm{o}(t,\sigma) = \Phi_\mathrm{b} - \Theta\beta(t)\Gamma_\mathrm{o}$$

$$\Phi_\mathrm{b} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \Theta = \begin{pmatrix} \theta_{q-1}^\mathrm{T} \\ \vdots \\ \theta_1^\mathrm{T} \\ \theta_0^\mathrm{T} \end{pmatrix}.$$

- Choose L_0 such that $F = \Phi_\mathrm{b} - L_0\Gamma_\mathrm{o}$ is Hurwitz
- Let $G(t,\theta) = L_0 - \Theta\beta(t)$, $H = \Gamma_\mathrm{o}$
  - The triplet $(F, G(\cdot,\theta), H)$ is not yet in *feedback form*.
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Non-minimal Internal Model Unit

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$$F_0 = \begin{pmatrix} -l_{q-1} & \cdots & -l_1 & -l_0 \\ 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \end{pmatrix} \quad G_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad H_0 = L_0'$$

$$F_1 = \begin{pmatrix} -l_{q-1}I_\rho & \cdots & -l_1 I_\rho & -l_0 I_\rho \\ I_\rho & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & I_\rho & 0 \end{pmatrix} \; G_1(t) = \begin{pmatrix} \beta(t) \\ 0 \\ \vdots \\ 0 \end{pmatrix} \; H_1(\theta) = \theta'$$

Usefulness of Weak Immersions

The triplet

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The canonical parameterization has been obtained via a weak immersion of the original exosystem.

$$\big(S(t,\sigma), R(t,\mu)\big) \overset{\text{strong}}{\longrightarrow} \big(\Phi_o(t,\theta), \Gamma_o\big) \overset{\text{weak}}{\longrightarrow} \big(F_{\rm im}, G_{\rm im}(t), H_{\rm im}(\theta)\big)$$

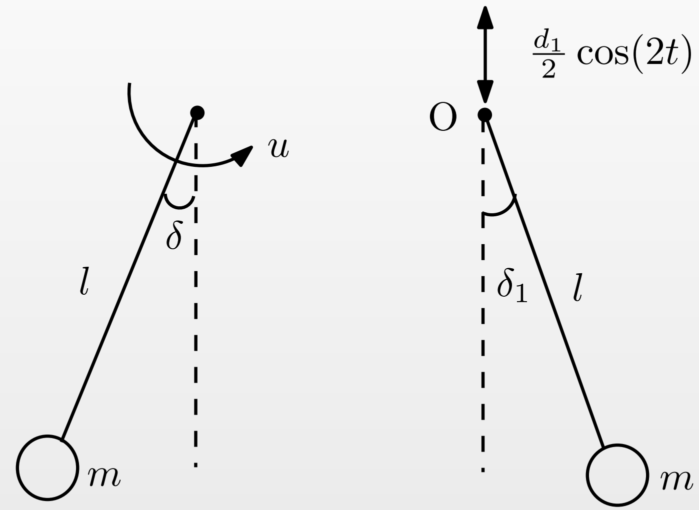
Synchronization of a Pendulum with a Mathieu System

Controlled pendulum:

$$\ddot{\delta} = -a\delta + bu$$

Vertically-oscillating pendulum:

$$\ddot{\delta}_1 = (-a - 2d\cos(2t))\delta_1$$

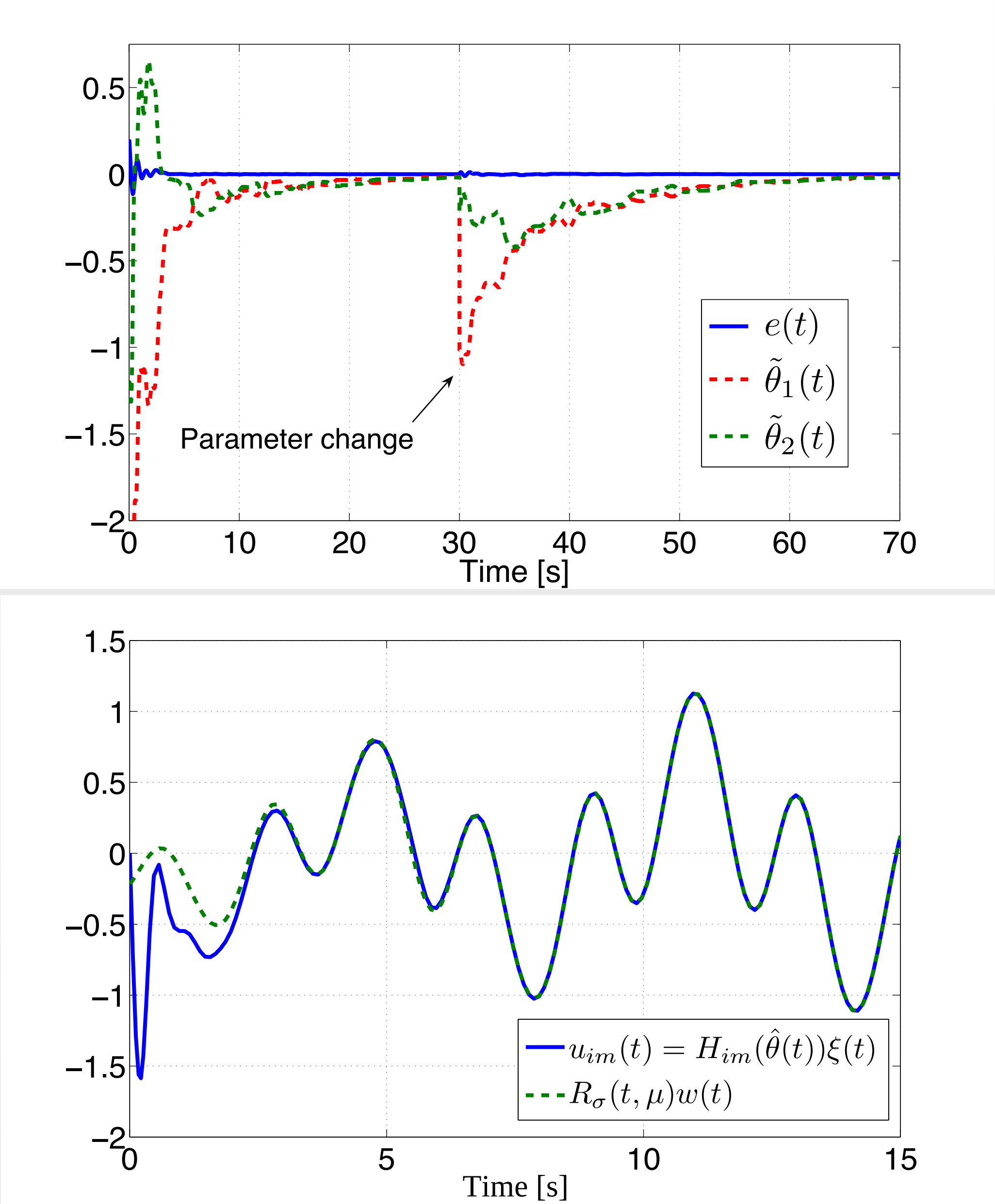


Exosystem

$$S(t,\sigma) = \begin{pmatrix} 0 & 1 \\ -a - 2d\cos(2t) & 0 \end{pmatrix}, \quad R_\sigma(t,\mu) = \begin{pmatrix} \frac{2}{b}d\cos(2t) & 0 \end{pmatrix}$$

strongly immersed in a 4-dim IM pair $(\Phi_o(t,\theta), \Gamma_o)$ in observer canonical form with new parameter vector $\theta = (a,\ d,\ a^2,\ ad)'$.

Simulation Results



Conclusions

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- Weaker detectability properties are related to the possibility of obtaining canonical realizations of periodic internal models to be used in certainty-equivalence design to deal with parameter uncertainty on the exosystem model.
- **Open problem:** It is not clear whether coordinate-free conditions can be found to check a priori the existence of a regular immersion.